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The Domination Number of the Co-prime Power Order Graph of a Finite Group

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Abstract:

Let G be a group with order greater than 2. Define the graph β_G by its vertex set G, where two distinct elements $x\ and\ y \in G$ are adjacent if and only if $\gcd(|x|,|y|) = p^n$, where p is a prime integer number that divides $|G|$, and $n \in \mathbb{Z}^+ \cup \{0\}$. In this study, we determine the dominating set and the dominating number of β_G when G is finite. We examine how the orders of elements influence adjacency and the effect properties of the greatest common divisors of element orders to characterize potential dominating sets. Our results provide explicit bounds and, in some finite groups, exact values for the dominating number.

Keywords: complete graph, coprime graph , co-prime order graph, co-prime power order graph, dominating set, dominating number.

1. INTRODUCTION

Suppose that G is a finite group with order greater than 1, Sattanathan et al. introduced the co-prime graph, which is represented by Γ_G as graph whose two distinct vertices $a, b \in \Gamma_G$ are adjacent if and only if $(|a|,|b|) = 1$ [1,2]. After that, generalized the co-prime graph into the co-prime order graph, which is denoted by $\Theta(G)$ such that two distinct vertices $a, b \in G$ are adjacent if and only if $\gcd(|a|,|b|) = 1\ or\ a\ prime\ number$ [3,4]. H. H. Mushatet & A.A Talebi generalized the co-prime order graph into the co-prime power order graph of a finite group G, which is represented by β_G is an undirected graph where vertices represent elements of G, such that two distinct vertices $x, y \in G$ are adjacent if and only if $\gcd(|x|,|y|) = p^n$,with a prime number p, and n is a non-negative integer[5]. In [5] they worked on planarity and the completeness of β_G of finite groups. After that, H. H. Mushatet & A.A Talebi investigated on complete and planar β_G structure of a finite group [5].

Clearly, the readers can see thst the researcher studied an important property of a co-prime power order graph β_G, specifically, the domination properties. This research is organised as follows: in Section 2, we introduce the preliminary definitions, propositions, lemmas, and theorems used throughout this paper. In section 3, we represented the domination of graph β_G . Finally, we introduced the conclusion of this study.

2. Basics and Notions

In this part, we introduce a few notions, definitions, and theorems that will be used in this study to determine the dominating number of graph β_G of the finite group G.

Definition 2.1[5]**:** Let G be a group with order $n > 2$, then the subset $\{x \in G||x| = p^m, with\ prime\ p\ , and\ m\ is\ non-negative\ integer\ number\ \}$ denoted by $P(G)$.

Definition2.2[5]**:** A $\phi - group$ is a group with size greater than 2 and each nontrivial element has a prime power order.

Definition 2.3[6 –11]**:**A graph Γ is termed a complete graph if and only if each pair vertices in $V(\Gamma)$ are neighbor.

Theorem 2.1[5]**:** A group G with $|G| > 2$ is $\phi - group$ if and only if The co-prime power order graph β_G is a complete graph.

Definition 2.4[11–15]**:** A dihedral group of order $2n$ denoted by D_n is a group that is defined as

$D_n =< a, b|a^n = b^2 = e, and\ bab = a^{-1}, n \in \mathbb{Z}^+ >.$

Definition 2.5[11–15]**:** A generalized Quaternion group denoted by Q_{4n} is a group that is defined as:

$Q_{4n} =< a, b|\ a^n = b^2,\ a^{2n} = b^4 = 1\ ,\ bab = a^{-1} >$, for $, n \in \mathbb{Z}^+$.

Theorem 2.2[5]**:** Let we have $G = C_n$. A graph β_G is complete if and only if $n = p^k, p\ is\ a\ prime\ integer\ , and\ k \geq 1$.

Theorem 2.3[5]**:** Suppose that $G = D_n$,then β_G is a complete graph if and only if $n = p^m,\ p\ is\ a\ prime\ number\ divides\ |G|\ , and\ m$ is a positive integer number.

Theorem 2.4[5]**:** Suppose that $G = Q_{4n}$. Then β_G is complete if and only if $n = 2^m,\ m \in \mathbb{Z}^+$.

Definition 2.6[6 –11]**:** A non-empty set $S \subseteq V(\Gamma)$ named clique in a graph Γ iff every two vertices in S are neighbor in graph Γ.

Definition 2.7[6 –11]**:** The maximum clique of a graph Γ named A clique number of a graph Γ, and it denoted by $\omega(\Gamma)$.

Definition 2.8[6 –11]**:** Suppose we have a graph Γ with vertex set $V(\Gamma)$ then the induced subgraph of subset $S \subseteq V(\Gamma)$ is denoted by Γ[S] when $x\ and\ y$ are neighbor in Γ iff $x\ and\ y$ are neighbor in Γ[S].

Definition 2.9[5]**:** Suppose that β_G is the co-prime power order graph of a group G then the induced subgraph on the vertex set $S \subseteq V(\beta_G)$ is denoted by β_G[S] when $x\ and\ y$ are adjacent in β_G iff $x\ and\ y$ are neighbor in β_G[S].

Definition 2.10[6 –11]**:** Let we have a graph Γ such that $V(\Gamma) = \coprod_{i=1}^{k} V_i(\Gamma)$, and $V_1(\Gamma), V_2(\Gamma), \dots, V_k(\Gamma)$ are distinct vertex sets , $k \in \mathbb{Z}^+$ then Γ is called $k-partite$ graph if and only if every edge have one end point in V_i , and other in V_j for $i \neq j$.

Theorem 2.5[5]**:** Suppose we have a group G, after that the induced subgraph $\beta_G[\mathrm{G} - \mathrm{P(G)}]$ is a null graph iff $|G| = p^{\alpha} q^{\gamma}$, for prime factors p, q, and $\alpha, \gamma \in \mathbb{Z}^+$.

Theorem 2.6[5]**:** Suppose that we have a graph β_G of a finite group G, then $\omega(\beta_G) = |P(G)| + 1$ if and only if $|G| = p^{\gamma_1} q^{\gamma_2}$, for distinct prime numbers p, q, and $\gamma_1, \gamma_2 \in \mathbb{Z}^+$.

Theorem 2.7[5]**:** Suppose that we have graph β_G of a finite group G, where $|G| = p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3}$, for distinct prime numbers p_1, p_2, p_3, and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}^+$. Then $\omega(\beta_G) = |P(G)| + 3$.

Definition 2.11[6 –11]**:** Suppose that we have Γ_G with a finite group G. Then a non-empty $D \subseteq V(\Gamma)$ named dominating set if $\forall x \in D$ there exist $y \in V(\Gamma), y \notin D$ when $x\ and\ y$ are neighbor.

Definition 2.12[6 –11]**:** A size of minimum dominating set of the co-prime graph Γ_G is known as dominating number and represented by $\gamma(\Gamma_G)$.

3. THE DOMINATING NUMBER OF GRAPH β_G OF A FINITE GROUP

In this part of study, we discuss some results on the dominating number of graph β_G of finite groups as $\mathbb{Z}_n$, D_n, and Q_{4n}. Also, we discuss some results on the dominating number of the induced subgraph $\beta_G[G - P(G)]$ with specific order of G.

Definition 3.1: Suppose that we have a graph β_G with group G, then a non-empty set $D \subseteq V(\beta_G)$ is called dominating set when any $x \in D$ there exist $y \in V(\beta_G), y \notin D$ implied $x\ and\ y$ be neighbor.

Definition 3.2: The dominating number of graph β_G with group G is smallest order of the dominating sets of $V(\beta_G)$, and it represented by $\gamma(\beta_G)$.

Lemma 3.1: Suppose that we have a graph β_G, and $D \subseteq V(\beta_G)$ is a domination set of β_G, when G be a $\phi - group$ then the maximum dominating set is $D = V(\beta_G) - \{v\}$, so the minimum dominating set is $D = \{u\}$, such that $u\ and\ v$ are arbitrary vertices in $V(\beta_G)$.

Proof: Let G be a $\phi - group$, then for every $x \in G$ we obtain $|x| = p^m, for$ prime $p, m \in \mathbb{Z}^+$. Therefore any $\forall\ x, y$ in $V(\beta_G)$ are neighbor. Hence a maximum dominating set of β_G is $G - \{v\}$, such that is an arbitrary vertex of $V(\beta_G)$ and adjacent with any vertex of $G - \{v\}$.

Now, we take $D = \{u\}$. Since G is a $\phi - group$, then each vertex $v \in G - D$ is adjacent to u. Hence $D = \{u\}$ is a minimum dominating set of β_G.

Theorem 3.2: Let we have a graph β_G of a finite group G, then β_G is a complete graph if and only if the dominating number $\gamma(\beta_G) = 1$.

Proof: Assume that $\gamma(\beta_G) = 1$, then the dominating set $D \subseteq V(\beta_G)$ has one vertex x, and every $y \in V(\beta_G) - D$ be adjacent with v. It follows that $\gcd(|x|, |y|) = p^k$, p is prime number, and $k \in \mathbb{Z}^+$. Hence β_G is a complete graph.

Now, let β_G be complete graph, then by[Theorem 2.1] we obtain G is a $\phi - group$. Clearly, we obtain the maximum dominating set $D \subseteq V(\beta_G)$ has one vertex [Lemma 3.1]. Therefore $\gamma(\beta_G) = 1$.

Theorem 3.3: Suppose that $G = \mathbb{Z}_n$, and we have the graph $\beta_{\mathbb{Z}_n}$, then the dominating number $\gamma\left(\beta_{\mathbb{Z}_n}\right) = 1$ iff $n = p^k$, p is prime, and $k \in \mathbb{Z}^+$.

Proof: Suppose that $\gamma\left(\beta_{\mathbb{Z}_n}\right) = 1$, then $\beta_{\mathbb{Z}_n}$ is complete graph [Theorem 3.2]. It follows that every two vertices $x\ and\ y \in V(\beta_{\mathbb{Z}_n})$ are adjacent. Thus, we obtain $\gcd(|x|, |y|) = p^m$, and p^m/n. Hence n is a prime power.

Now, let we have $n = p^k$, then $\beta_{\mathbb{Z}_n}$ is a complete graph [Theorem 2.2]. Therefore $\gamma\left(\beta_{\mathbb{Z}_n}\right) = 1$[Theorem 3.2].

Example 3.1: Let $G = \mathbb{Z}_8 = \mathbb{Z}_{2^3} = \{0,1,2,3,4,5,6,7\}$, then $|1| = |3| = |5| = |7| = 2^3$, $|2| = |6| = 2^2$, and $|4| = 2$. By [Theorem 2.2] we obtain that β_G is a complete graph (Figure 3.1). It follows that the minimum dominating set $D \subseteq V(\beta_{\mathbb{Z}_8})$ is $\{u\}$ and $u \in V(\beta_{\mathbb{Z}_8})$, and the maximum dominating set $D \subseteq V(\beta_{\mathbb{Z}_8})$ is $V(\beta_{\mathbb{Z}_8}) - v$, and $v \in V(\beta_{\mathbb{Z}_8})$. It follows that $\gamma\left(\beta_{\mathbb{Z}_8}\right) = 1$

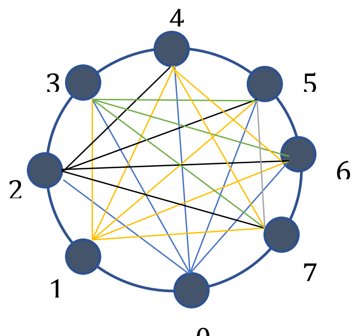

Figure 3.1: Complete graph of $\mathbb{Z}_8$

Theorem 3.4: Let $G = D_n$, and we have the graph β_{D_n}, then the dominating number $\gamma\left(\beta_{D_n}\right) = 1$ iff $n = p^m$, p is prime, and $m \in \mathbb{Z}^+$.

Proof: Assume we have $\gamma(\beta_{D_n}) = 1$, then β_{D_n} is complete graph [Theorem 3.2]. It follows that every two vertices $x\ and\ y \in V(\beta_{D_n})$ are neighbor. Hence, we have $\gcd(|x|, |y|) = p^k$, and p^k/n. Hence n is a prime power.

Now, suppose we have $n = p^m$, then β_{D_n} is a complete graph [Theorem 2.2]. Therefore $\gamma(\beta_{D_n}) = 1$[Theorem 3.2].

Example 3.2 : Let $G = D_{2^3} = D_8 = \{e, a, a^2, \dots, a^7, b, ab, a^2b, \dots, a^7b\}$, then we see that $|a| = |a^3| = |a^5| = |a^7| = 2^3$, $|a^2| = |a^6| = 2^2$, $|a^4| = 2$, and $|b| = |ab| = |a^2b| = \cdots = |a^7b| = 2$. By [Theorem 2.3] we obtain that β_{D_8} is a complete graph (Figure 3.2). It follows that the minimum dominating set $D \subseteq V(\beta_{D_8})$ is $\{u\}$ and $u \in V(\beta_{D_8})$, and the maximum dominating set $D \subseteq V(\beta_{D_8})$ is $V(\beta_{D_8}) - v$, and $v \in V(\beta_{D_8})$. It follows that $\gamma(\beta_{D_8}) = 1$.

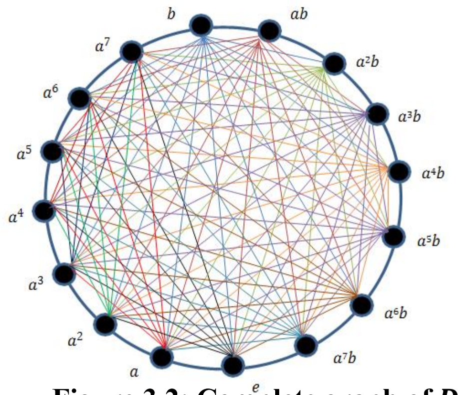

Figure 3.2: Complete graph of D_8

Theorem 3.5: Let $G = Q_{4n}$, and suppose we have the graph $\beta_{Q_{4n}}$, then the dominating number $\gamma(\beta_{Q_{4n}}) = 1$ if and only if $n = 2^m, m \in \mathbb{Z}^+$.

Proof: Let $\gamma(\beta_{Q_{4n}}) = 1$, then by [Theorem 3.2] we obtain that $\beta_{Q_{4n}}$ is complete graph. It follows that for any two vertices $x\ and\ y \in V(\beta_{Q_{4n}})$ are adjacent, and $n = 2^m$[Theorem 2.4].

Now, we suppose that $n = 2^m$, then $\beta_{Q_{4n}}$ is a complete graph [Theorem 2.4]. Therefore $\gamma(\beta_{Q_{4n}}) = 1$[Theorem 3.2].

Example 3.3: Let $G = Q_{16} = Q_{4(4)} = \{e, a, a^2, \dots, a^7, b, ab, a^2b, \dots, a^7b\}$. From the elementary group theory, we know that if $a^i \in Q_{4n}$, then $|a^i| = \frac{2n}{\gcd(i,2n)}, n \in \mathbb{Z}^+$. It follows that $|a^i| = \frac{8}{\gcd(i,8)} = 2^m, m \in \mathbb{Z}^+$. Also $|a^ib| = 4 = 2^2$. By [Theorem 2.3] we obtain that $\beta_{Q_{16}}$ is a complete graph (Figure 3.2). It follows that the minimum dominating set $D \subseteq V(\beta_{Q_{16}})$ is $\{u\}$ and $u \in V(\beta_{Q_{16}})$, and the maximum dominating set $D \subseteq V(\beta_{Q_{16}})$ is $V(\beta_{Q_{16}}) - v$, and $v \in V(\beta_{Q_{16}})$. It follows that $\gamma(\beta_{Q_{16}}) = 1$.

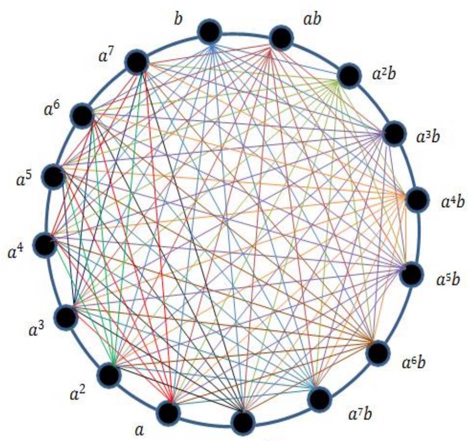

Figure 3.3: Complete graph of Q_{16}

Theorem 3.6: Suppose we have a graph β_G of a finite group G, when $|G| = p^{\alpha}q^{\delta}$, it implies $\gamma(\beta_G) = 1$, and $\gamma(\beta_G[G - P(G)]) = |[G - P(G)]|$ p, q are prime numbers, and $\alpha, \delta \in \mathbb{Z}^+$.

Proof: Suppose that $|G| = p^{\alpha}q^{\delta}$, then $\beta_G[G - P(G)]$ be null graph, and $\forall x, y \in V(\beta_G[G - P(G)])$ are not adjacent [Theorem 2.5]. Furthermore that, we obtain for every vertex $u \in V(\beta_G)$, $u \notin V(\beta_G[G - P(G)])$ has order $p^{\alpha'}$ or $q^{\delta'}$, and u adjacent with some vertices of a null graph $\beta_G[G - P(G)]$. Since the identity element of group G is adjacent with other vertices of β_G. Therefore, the minimum dominating set of graph β_G is $\{e\}$.

Hence $\gamma(\beta_G) = 1$. Finally, every vertex $v \in V(\beta_G[G - P(G)])$ is dominated with itself, and the minimum dominating set of $\beta_G[G - P(G)]$ is equal to $\{G - P(G)\}$. It follows that $\gamma(\beta_G[G - P(G)]) = |[G - P(G)]|$.

Example 3.4: Suppose that $G = \mathbb{Z}_{10}$, then $|1| = |3| = |7| = |9| = 10$, $|2| = |4| = |6| = |8| = 5$, $|0| = 1$, and $|5| = 2$. It follows that $P(G) = \{0,2,4,5,6,8\}$. $\mathbb{Z}_{10} - P(\mathbb{Z}_{10}) = \{1,3,7,9\}$, and the order of every two vertices x and $y \in \mathbb{Z}_{10} - P(\mathbb{Z}_{10})$ are not adjacent. Hence $\beta_{\mathbb{Z}_{10}}[\mathbb{Z}_{10} - P(\mathbb{Z}_{10})]$ is the null graph (Figure 3.4, 3.5). It follows that the minimum dominating set $D \subseteq V(\beta_{\mathbb{Z}_{10}})$ is $\{0\}$. It follows that $\gamma(\beta_{\mathbb{Z}_{10}}) = 1$. Now the minimum dominating set $D \subseteq \beta_{\mathbb{Z}_{10}}[\mathbb{Z}_{10} - P(\mathbb{Z}_{10})]$ is equal to the maximum dominating set $D \subseteq \beta_{\mathbb{Z}_{10}}[\mathbb{Z}_{10} - P(\mathbb{Z}_{10})]$ such that $D = \{1,3,7,9\}$. Hence $\gamma(\beta_{\mathbb{Z}_{10}}[\mathbb{Z}_{10} - P(\mathbb{Z}_{10})]) = 4 = |\mathbb{Z}_{10} - P(\mathbb{Z}_{10})|$.

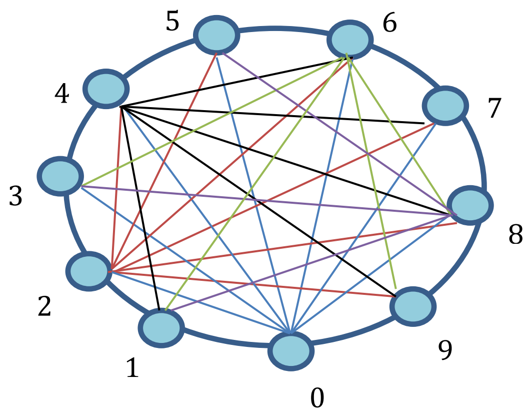

Figure 3.4: The graph of $\beta_{\mathbb{Z}_{10}}$

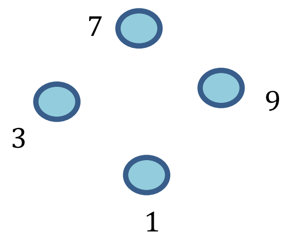

Figure 3.5: $\beta_{\mathbb{Z}_{10}}[\mathbb{Z}_{10} - P(\mathbb{Z}_{10})]$

CONCLUSION

In this paper, we obtained a few results on graph β_G of finite group. These results focus on a domination number of a minimum domination set of order 1 and a maximum domination set of order $|P(G)| - 1$, when G be a ϕ −group. Furthermore, we found the dominating number of the graph β_G of a specific groups as $\mathbb{Z}_n$, D_n, Q_{4n}, and some induced graph of the co-prime power order graph which the results found $(\beta_{\mathbb{Z}_n}) = 1$ if and only if $n = p^k$, $\gamma(\beta_{D_n}) = 1$ if and only if $n = p^m$, $\gamma(\beta_{Q_{4n}}) = 1$ if and only if $n = 2^m$, when $|G| = p^\alpha q^\delta$, then $\gamma(\beta_G) = 1$, and $\gamma(\beta_G[G - P(G)]) = |[G - P(G)]|$.

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