Cartesian Product of Two Modular Fuzzy Metric Spaces

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Abstract:

This paper investigates the properties of modular fuzzy metric spaces and their Cartesian products. a
We begin with the basic definitions and properties of modular metrics and modular fuzzy metrics.
The main results focus on establishing that the Cartesian product of two modular fuzzy metric spacesu
isf itself modular fuzzyt metricq space. We prove that convergence and Cauchy sequences in the product
space correspond to the sequences in the component spaces. Furthermore, the completeness property is
preserved under Cartesian products. These results provide a rigorous framework for further studies in
fuzzy analysis and its applications in metric-based uncertainty modeling.

Keywords: Modular fuzzy metric space ,Cartesian product, Convergence, Completeness , Cauchy
sequences

1- Introduction

The concept of modular space was introduced by Nakano [8] in1950. The theory of fuzzy sets was first introduced by Zadeh [17] in 1965 as a generalisation of the concept of a set. Kramosil and Michalek [11] in 1975 introduced the concept of fuzzy metric spaces, independent of the definitions of metric spaces and fuzzy sets. Latar, George and Veeramani [6] introduced some results on the notion M.J of a fuzzy metric space. In 2010, Chistyakov [4 ,5] introduced modular metric spaces, which are metric spaces equipped with a modular metric. Thereafter, many generalizations, extension of these celebrated results enriched the theory of modular metric spaces by many authors. The concept of fuzzy modular space was introduced by Y. Shen and W. Chen [15] studied the properties in 2013. The definition of the Cartesian product of two fuzzy modular spaces was introduced by Noor F. Al-Mayahi and Al-ham S. Nief [10] in 2019, and some results about it were proved. In 2020, Kadim and M. J. Mohammed [13] defined the Cartesian product of intuitionistic fuzzy modular spaces. In 2022, Kerim et al. [12] introduced a new space, the modular fuzzy metric space, and studied some fixed-point results in this space. Subsequently, several other mathematicians discussed definitions of modular fuzzy metric spaces in their works; for example, see the references [2,3,7,9] for alternative definitions. In 2025, L.A. Zarzour and M. J. Mohammed [16] presented the Cartesian product of intuitionistic fuzzy rectangular n-normed spaces and proved some related theorems. In this paper, we

introduce the concept of the Cartesian product in a modular fuzzy metric space, then show that the Cartesian product of two modular fuzzy metric spaces is also a modular fuzzy metric space, and present some theorems about it.

2- Preliminaries

In this section, we will recall some definitions that are used in this paper.

Definition 2.1 [4]

A modular metric on a nonempty set $Ÿ$ is a function $Ѓ : (0, \infty) \times Ÿ \times Ÿ \rightarrow [0, \infty)$

that will be written as $Ѓ_\delta (ï, n) = Ѓ(\delta, ï, n)$; for all $ï, n, l \in Ÿ$ and for all $\delta, \sigma > 0$,

satisfy the following three conditions:

1. $Ѓ_\delta (ï, n) = 0 \Leftrightarrow ï = n, \forall \delta > 0$ and $ï, n \in Ÿ$

2. $Ѓ_\delta (ï, n) = Ѓ_\delta (n, ï), \forall \delta > 0$ and $ï, n \in Ÿ$

3. $Ѓ_{\delta+\sigma} (ï, n) \leq Ѓ_\delta (ï, l) + Ѓ_\sigma (l, n), \forall \delta, \sigma > 0$ and $ï, n, l \in Ÿ$.

Definition 2.2 [5]

A modular $Ѓ$ on $Ÿ$, a sequence $\{ï_n\}$ in $Ÿ_Ѓ$ is said to be modular

convergent to if and only if $\lim_{n \rightarrow \infty} Ѓ_\delta(ï_n, ï) = 0$, for all $\delta > 0$. $ï \in Ÿ_Ѓ$

Definition 2.3 [5]

Let $Ѓ$ be a modular on $Ÿ$, a sequence $\{ï_n\}$ in $Ÿ_Ѓ$ is said to be modular Cauchy

if and only if , for all $\delta > 0$. $\lim_{n,m \rightarrow \infty} Ѓ_\delta (ï_n, ï_m) = 0$

Definition 2.4 [17]

A modular space $Ÿ_Ѓ$ is said to be modular complete if each Cauchy sequence in $Ÿ_Ѓ$ is modular

   Convergent.

Definition 2.5[16]

Let $Ÿ$ be any set. A fuzzy set V in $Ÿ$ is a function with domain $Ÿ$ and values in [0,1].

Definition 2.6[14]

A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$. An operator $*$ is a continuous t-norm

if   for all $p, q, r, m \in [0,1]$ satisfy:

(a) $p * q = q * p.$

(b) $(p * q) * s = p * (q * s).$

(c) $p * 1 = p$.

(d) If $p \leq s$, and $q \leq m$, then $p * q \leq s * m$.

Theorem 2.7 [1]

If $*$ is a continuous t-norm, then:

(1) $1 * 1 = 1$

(2) $0 * 1 = 0$

(3) $0 * 0 = 0$

(4) $a * a \leq a, \forall a \in I = [0,1]$

(5) If $a \leq c$ and $b \leq d$, then $a * b \leq c * d$ for all $a, b, c, d \in I = [0,1]$.

Definition 2.8 [11]

The triple ($\ddot{Y}, U, *$) is said to be fuzzy metric space (In short, FMS)

if Y is an arbitrary set, $*$ a is continuous t-norm and U is a fuzzy set on

$\ddot{Y} \times \ddot{Y} \times (0, \infty) \rightarrow [0,1]$ for all $ï, ℩, ℓ \in \ddot{Y}$ and $t, s > 0$, satisfying the following conditions:

(a) $U(ï, ℩, t) > 0,$

(b) $U(ï, ℩, t) = 1$ if and only if $ï = ℩,$

(c) $U(ï, ℩, t) = U(℩, ï, t),$

(d) $U(ï, ℩, t + s) \geq U(ï, ℓ, t) * U(ℓ, ℩, s)$,

(e) $U(ï, ℩, .): (0, \infty) \rightarrow [0,1]$ is   continuous.

Example 2.9 [6]

If ($\ddot{Y}, d$) be a metric spase. Define $\alpha * \beta = \min\{\alpha, \beta\}$ and

$U(ï, ℩, t) = \dfrac{t}{t + d(x,y)}$, $\forall ï, ℩, ℓ \in Y$ and $t > 0$. Then ($\ddot{Y}, U, *$) is a FMS.

Definition 2.10 [6]

Let $(\ddot{Y}, U, *)$ be FMS:

(1) A sequence $\{\ddot{\imath}_n\}$ in Y is convergent to $\ddot{\imath} \in \ddot{Y}$ if $\lim\limits_{n \to \infty} U(\ddot{\imath}_n, \ddot{\imath}, t) = 1$, $\forall\, t > 0$.

A sequence $\{\ddot{\imath}_n\}$ in $\ddot{Y}$ is called Cauchy sequence if $\lim\limits_{n \to \infty} U(\ddot{\imath}_n, \ddot{\imath}_m, t) = 1, \forall\, t > 0.$ (2)

(3) A FMS $(\ddot{Y}, U, *)$ is called complete if each Cauchy sequence in $\ddot{Y}$ is convergent sequence.

Definition 2.11 [12]

The triple $(\ddot{Y}, \dot{N}_\delta, *)$ is said to be a modular fuzzy metric space (In short, MFMS)

if $\ddot{Y}$ is an arbitrary set, $*$ a is continuous t-norm and $\dot{N}_\delta$ is a fuzzy set

on $(0, \infty) \times \ddot{Y} \times \ddot{Y} \times (0, \infty) \to [0,1]$ for all $\ddot{\imath}, n, l \in \ddot{Y}$

satisfying the following condition:

(1) $\dot{N}_\delta(\ddot{\imath}, n, t) > 0$, $\forall\, \delta, t > 0$.

(2) $\dot{N}_\delta(\ddot{\imath}, n, t) = 1 \Leftrightarrow \ddot{\imath} = n$, $\forall\, \delta, t > 0$

(3) $\dot{N}_\delta(\ddot{\imath}, n, t) = \dot{N}_\delta(n, \ddot{\imath}, t), \forall\, \delta, t > 0.$

(4) $\dot{N}_{\delta+\sigma}(\ddot{\imath}, n, t + s) \geq \dot{N}_\delta(\ddot{\imath}, l, t) * \dot{N}_\sigma(l, \ddot{\imath}, s)$, $\forall\, \delta, \sigma, t, s > 0.$

(5) $\dot{N}_\delta(\ddot{\imath}, n,): (0, \infty) \to [0,1]$ is continuous for all $\delta > 0$.

Here, $\dot{N}_\delta$ is called a modular fuzzy metric.

Example 2.12 [12]

Let $(\ddot{Y}, \acute{\Gamma}_\delta)$ be a modular metric space. Define $\alpha * \beta = \alpha.\beta$ for

all $\alpha, \beta \in [0,1]$ and $\dot{N}_\delta: (0, \infty) \times \ddot{Y} \times \ddot{Y} \times (0, \infty) \to [0,1]$ define by

$$\dot{N}_\delta(\ddot{\imath}, n, t) = e^{-\frac{\acute{\Gamma}_\delta(\ddot{\imath}, n)}{t}} \quad .$$

Then $(\ddot{Y}, \dot{N}_\delta, *)$ is a MFMS.

Definition 2.13 [12]

Let $(\ddot{Y}, \dot{N}_\delta, *)$ be a MFMS:

$\{\ddot{\imath}_n\}$ in $\ddot{Y}$ is convergent to an element $\ddot{\imath} \in \ddot{Y}$ if $\lim\limits_{n \to \infty} \dot{N}_\delta(\ddot{\imath}_n, \ddot{\imath}, t) = 1$ 1) sequence

, $\forall\, \delta, t > 0$

A sequence $\{\ddot{\imath}_n\}$ in $\ddot{Y}$ is called Cauchy sequence if $\lim\limits_{n,m \to \infty} \dot{N}_\delta(\ddot{\imath}_n, \ddot{\imath}_m, t) = 1$ 2)

$, \forall\ \delta, t > 0.$

3) A MFMS ($\ddot{Y}, \dot{N}_\delta$,$*$) is called complete if every Cauchy sequence in $\ddot{Y}$ is also a

   convergent   sequence.

4) A MFMS($\ddot{Y}, \dot{N}_\delta$,$*$) is said to be compact if every sequence in $\ddot{Y}$ has at least one

   Convergent subsequence.

3- Main Resuit

In this section, we introduce the concept of the Cartesian product in modular fuzzy metric space, then

we prove that the cartesian product of two modular fuzzy metric spaces is also modular fuzzy metric

space Finally, we prove completeness of the cartesian product of two modular fuzzy metric spaces

complete modular fuzzy metric space.

Definition 3.1 :

Let $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$, $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ be   two modular fuzzy metric spaces . The

Cartesian product   of $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$ and $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ is the product space, $\dot{N}_\delta$

$(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$ where $\ddot{Y}_1 \times \ddot{Y}_2$ is   the Cartesian product of the sets $\ddot{Y}_1$ and $\ddot{Y}_2$

 is a function from $(0, \infty)\ \times (\ddot{Y}_1 \times \ddot{Y}_2\ ) \times (0, \infty)\ \times (0, \infty)\ \times\ (\ddot{Y}_1 \times\ \ddot{Y}_2\ ) \times (0, \infty)$ into $[0,1]$

and $\dot{N}_\delta\ \ \left((\dot{n}_1, \dot{n}_2),(\dot{m}_1, \dot{m}_2), t\ \right) =\ \ \dot{N}_{\delta 1}\ (\dot{n}_1,\ \dot{m}_1,\ t) *\ \dot{N}_{\delta 2}\ (\dot{n}_2,\ \dot{m}_2,\ t)$

 for all $(\dot{n}_1, \dot{n}_2), (\dot{m}_1, \dot{m}_2) \in \ddot{Y}_1 \times\ \ddot{Y}_2 \forall t, \delta\ >\ 0.$

Theorem 3.2 :

Let $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$ and $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ be two MFMS s.   Then $(\ddot{Y}_1 \times\ \ddot{Y}_2, \dot{N}_\delta, *)$ is MFMS.

Proof:

Let $(\dot{n}_1, \dot{n}_2), (\dot{m}_1, \dot{m}_2) \in \ddot{Y}_1 \times\ \ddot{Y}_2$

1) since $\dot{N}_{\delta 1}\ \ (\dot{n}_1,\ \dot{m}_1,\ t) > 0$ and $\dot{N}_{\delta 2}\ \ (\ \dot{n}_2, \dot{m}_2,\ t) > 0\ \forall t > 0$ and $\delta_1, \delta_2 > 0$, then

$\dot{N}_\delta\ \ ((\dot{n}_1, \dot{n}_2),(\dot{m}_1, \dot{m}_2), t\ )\ \ =\ \ \dot{N}_{\delta 1}\ (\dot{n}_1, \dot{m}_1,\ t)\ \ *\ \dot{N}_{\delta 2}\ (\dot{n}_2,\ \dot{m}_2,\ t)\ \ >\ \ 0$

2) $\dot{N}_{\delta 1}\ \ (\dot{n}_1, \dot{m}_1, t) = 1$ if and only   if $\dot{n}_1 = \dot{m}_1$, also

$\dot{N}_{\delta 2}\ (\dot{n}_2, \dot{m}_2, t) = 1$ if   and only   if $\dot{n}_2 =\ \dot{m}_2.$

Then $\dot{N}_{\delta 1}\ \ (\dot{n}_1, \dot{m}_1, t)\ \ *\ \dot{N}_{\delta 2}\ \ (\dot{n}_2, \dot{m}_2, t) = 1 \Leftrightarrow (\dot{n}_1, \dot{n}_2) = (\dot{m}_1, \dot{m}_2).$ Hence

$\dot{N}_\delta\big((\dot{n}_1,\dot{n}_2),(\dot{m}_1,\dot{m}_2),t\big) = 1 \Leftrightarrow (\dot{n}_1,\dot{n}_2) = (\dot{m}_1,\dot{m}_2) = 0 \ \forall t > 0 \text{ and } \delta > 0.$

3) since $\dot{N}_{\delta 1}(\dot{n}_1,\dot{m}_1, t) = \dot{N}_{\delta_2}(\dot{m}_1,\dot{n}_1,t)$, $\dot{N}_{\delta 2}(\dot{n}_2, \dot{m}_2, t) = \dot{N}_{\delta 2}(\dot{m}_2,\dot{n}_2, t)$

$\qquad \forall t , \delta_1 , \delta_2 > 0$, then

$\dot{N}_\delta\big((\dot{n}_1,\dot{n}_2),(\dot{m}_1,\dot{m}_2),t\big) = \dot{N}_{\delta 1}(\dot{n}_1,\dot{m}_1, t) * \dot{N}_{\delta 2}(\dot{n}_2, \dot{m}_2, t)$

$\dot{N}_{\delta_2}(\dot{m}_1,\dot{n}_1,t) * \dot{N}_{\delta 2}(\dot{m}_2,\dot{n}_2,t) = \dot{N}_\delta\big((\dot{m}_1,\dot{m}_2),(\dot{n}_1,\dot{n}_2),t\big)$

for all $(\dot{n}_1,\dot{n}_2),(\dot{m}_1,\dot{m}_2) \in \ddot{Y}_1 \times \ddot{Y}_2 \ \forall t > 0 \text{ and } \delta > 0$

4) Since $(\ddot{Y}_1,\dot{N}_{\delta 1},*)$, $(\ddot{Y}_2,\dot{N}_{\delta 2},*)$ are two MFMS s.

$\dot{N}_{\delta_1 + \sigma_1}(\dot{n}_1, \dot{m}_1, t+s) \geq \dot{N}_{\delta_1}(\dot{n}_1,\dot{p}_1,t) * \dot{N}_{\sigma_1}(\dot{p}_1, \dot{m}_1,s)$

$\dot{N}_{\delta_2 + \sigma_2}(\dot{n}_2, \dot{m}_2, t+s) \geq \dot{N}_{\delta_2}(\dot{n}_2,\dot{p}_2,t) * \dot{N}_{\sigma_2}(\dot{p}_2, \dot{m}_2,s)$

for all $(\dot{n}_1,\dot{n}_2),(\dot{m}_1,\dot{m}_2)$, $(\dot{p}_1,\dot{p}_2) \in \ddot{Y}_1 \times \ddot{Y}_2, t, s, \delta, \sigma > 0$.

$\qquad \dot{N}_{\delta+\sigma}\big((\dot{n}_1,\dot{n}_2),(\dot{m}_1,\dot{m}_2),t+s\big) = \dot{N}_{\delta_1 + \sigma_1}(\dot{n}_1,\dot{m}_1, t+s) * \dot{N}_{\delta_2 + \sigma_2}(\dot{n}_2, \dot{m}_2, t+s)$

$\qquad \geq \big[\dot{N}_{\delta_1}(\dot{n}_1,\dot{p}_1,t) * \dot{N}_{\sigma_1}(\dot{p}_1, \dot{m}_1,s)\big] * \big[\dot{N}_{\delta_2}(\dot{n}_2,\dot{p}_2,t) * \dot{N}_{\sigma_2}(\dot{p}_2, \dot{m}_2,s)\big]$

$\qquad \geq \big[\dot{N}_{\delta_1}(\dot{n}_1,\dot{p}_1,t) * \dot{N}_{\delta_2}(\dot{n}_2,\dot{p}_2,t)\big] * \big[\dot{N}_{\sigma_1}(\dot{p}_1, \dot{m}_1,s) * \dot{N}_{\sigma_2}(\dot{p}_2, \dot{m}_2,s)\big]$

$\qquad \geq \dot{N}_{\delta+\sigma}\big((\dot{n}_1,\dot{n}_2),(\dot{p}_1,\dot{p}_2),t\big) * \dot{N}_{\delta+\sigma}\big((\dot{p}_1,\dot{p}_2),(\dot{m}_1,\dot{m}_2),s\big)$

5) since $\dot{N}_{\delta 1}(\dot{n}_1, \dot{m}_1, t): (0,\infty) \to [0,1]$ is continuous and

$\qquad \dot{N}_{\delta 2}(\dot{n}_2, \dot{m}_2, t) : (0,\infty) \to [0, 1]$ is continuous

$\dot{N}_\delta\big((\dot{n}_1,\dot{n}_2),(\dot{m}_1,\dot{m}_2),t\big): (0,\infty) \to (0, 1]$ is continuous

Hence $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta,*)$ is MFMS

Theorem 3.3 :

Let $\{\gamma_n\}$ be a sequence in MFMS $(\ddot{Y}_1,\dot{N}_{\delta 1},*)$

converge to γ in $\ddot{Y}_1$ and $\{\acute{\alpha}_n\}$ is a sequence in MFMS

$(\ddot{Y}_2,\dot{N}_{\delta 2},*)$ converge to $\acute{\alpha}$ in $\ddot{Y}_1$. Then $\{(\gamma_n,\acute{\alpha}_n)\}$ is a sequence

in MFMS $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta,*)$ converge to $(\gamma,\acute{\alpha})$ in $\ddot{Y}_1 \times \ddot{Y}_2$.

Proof:

To prove that sequence $\{(\gamma_n,\acute{\alpha}_n)\}$ in $\ddot{Y}_1 \times \ddot{Y}_2$ converges to $(\gamma,\acute{\alpha})$

We show that $\lim\limits_{n\to\infty} \dot{N}_\delta\big((\gamma_n,\acute{\alpha}_n),(\gamma,\acute{\alpha}),t\big) = 1$

By theorem 3.2 we have $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$ is MFMS.

Since $\{\gamma_n\}$ is a sequence in $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$ convergence to γ

Then $\lim\limits_{n\to\infty} \dot{N}_{\delta 1}(\gamma_n, \gamma, t) = 1.$

Since $\{\acute{\alpha}_n\}$ be a sequence in $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ convergence to $\acute{\alpha}$

Then $\lim\limits_{n\to\infty} \dot{N}_{\delta 2}(\acute{\alpha}_n, \acute{\alpha}, t) = 1.$ Then

$$\lim_{n\to\infty} \dot{N}_\delta\big((\gamma_n, \acute{\alpha}_n), (\gamma, \acute{\alpha}), t\big) = \lim_{n\to\infty} \dot{N}_{\delta 1}(\gamma_n, \gamma, t) * \lim_{n\to\infty} \dot{N}_{\delta 2}(\acute{\alpha}_n, \acute{\alpha}, t)$$

$$= \quad 1 * 1 = 1$$

Thus $\{(\gamma_n, \acute{\alpha}_n)\}$ converges to $(\gamma, \acute{\alpha})$.

Theorem 3.4 :

Let $\{\gamma_n\}$ be a Cauchy sequence in MFMS $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$

converge to γ in $\ddot{Y}_1$ and $\{\acute{\alpha}_n\}$ is a Cauchy sequence in MFMS

$(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ converge to $\acute{\alpha}$ in $\ddot{Y}_1$. Then $\{(\gamma_n, \acute{\alpha}_n)\}$ is a Cauchy sequence

in MFMS $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$.

Proof:

By theorem 3.2 $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$ MFMS

Since $\{\gamma_n\}$ be a Cauchy sequence in MFMS $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$

Then $\lim\limits_{n,m\to\infty} \dot{N}_{\delta 1}(\gamma_n, \gamma_m, t) = 1$

Since $\{\acute{\alpha}_n\}$ be a Cauchy sequence in MFMS $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$

Then $\lim\limits_{n,m\to\infty} \dot{N}_{\delta 2}(\acute{\alpha}_n, \acute{\alpha}_m, t) = 1$

$$\lim_{n,m\to\infty} \dot{N}_\delta\big((\gamma_n, \acute{\alpha}_n), (\gamma_m, \acute{\alpha}_m), t\big) = \lim_{n\to\infty} \dot{N}_{\delta 1}(\gamma_n, \gamma_m, t) * \lim_{n\to\infty} \dot{N}_{\delta 2}(\acute{\alpha}_n, \acute{\alpha}, t)$$

$$= \quad 1 * 1 = 1$$

Thus $\{\big((\gamma_n, \acute{\alpha}_n)\big)\}$ is a Cauchy sequence in $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$

Theorem 3.5 :

If $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$ and $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ are complete MFMSs

then $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$ is complete MFMS

Proof:

Let $\{(\gamma_n, \acute{\alpha}_n)\}$ be a Cauchy sequence in $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$

Since $(\ddot{Y}_1, \dot{N}_{\delta 1}, *)$ and $(\ddot{Y}_2, \dot{N}_{\delta 2}, *)$ are complete modular fuzzy metric

spaces

Then $\exists \gamma$ in $\ddot{Y}_1$ and $\acute{\alpha}$ in $Y \ni \{\gamma_n\}$ convergent to x

and $\{\acute{\alpha}_n\}$ convergent to $\acute{\alpha}$.

So $\lim\limits_{n \to \infty} \dot{N}_{\delta 1}(\gamma_n, \gamma, t) = 1$ and $\lim\limits_{n \to \infty} \dot{N}_{\delta 2}(\acute{\alpha}_n, \acute{\alpha}, t) = 1$

Now

$$\lim\limits_{n \to \infty} \dot{N}_\delta\big((\gamma_n, \acute{\alpha}_n), (\gamma, \acute{\alpha}), t\big) = \lim\limits_{n \to \infty} \dot{N}_{\delta 1}(\gamma_n, \gamma, t) * \lim\limits_{n \to \infty} \dot{N}_{\delta 2}(\acute{\alpha}_n, \acute{\alpha}, t)$$

$$= 1 * 1 = 1$$

Then $\{\{(\gamma_n, \acute{\alpha}_n)\}\}$ convergent to $(\gamma, \acute{\alpha})$ in $\ddot{Y}_1 \times \ddot{Y}_2$.

Hence $(\ddot{Y}_1 \times \ddot{Y}_2, \dot{N}_\delta, *)$ is complete MFMS.

Conclusion

In this study, we extended the theory of modular fuzzy 9metric2 spaces by exploring their Cartesian products. We demonstrated that the product of two modular fuzzy metric spaces preserves key properties, including convergence, Cauchy sequences, and completeness. These findings enhance the understanding of modular fuzzy structures and provide a solid foundation for future research in fuzzy analysis, decision-making models, and uncertainty quantification. The results are significant for both theoretical developments and practical applications in fields where fuzziness and metrics intersect.

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