Journal of Education for Pure Science
Vol.16, No. 1 (2026)
DOI: https://doi.org/10.32792/jeps.v16i1.829

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On Transform-Enhanced Iterative Methods for Nonlinear Fractional Partial Differential Equations

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Received 10 / 11 /2025, Accepted 5 / 2 /2026, Published 1/ 3 /2026

Abstract:

This paper presents two efficient methods for obtaining approximate solutions to fractional partial differential equations utilizing the Caputo approach. The Yasser–Jassim transform is the most important step. It changes the original fractional model into a simpler one in a different domain. Then we use the inverse transform to obtain a domain version that can be used for iteration. In the first scheme, this representation is used with the homotopy perturbation method to obtain a series solution that handles nonlinear terms consistently. The second scheme uses the transform and the Hussein–Jassim iterative method together. This makes it easy to calculate the series coefficients one step at a time. A number of linear and nonlinear test problems demonstrate how well the proposed methods perform. The numbers show that adding the transform does not affect the rate of convergence of the iterative method. Instead, it makes the math easier and helps you find the answer faster.

Keywords: Caputo fractional derivative, fractional partial differential equations, homotopy perturbation method, Hussein–Jassim method, Yasser–Jassim transform, and nonlinear models.

1-Introduction

Fractional calculus has become an important tool for modeling processes that have memory and behavior that isn't local. This has led to a lot of research on fractional differential equations in both theoretical and practical settings [1–10]. Analytical and semi-analytical methods continue to be preferred for these models because they can quickly produce series solutions without having to break them down into smaller parts [2,3,5–7]. The homotopy perturbation method (HPM) is a well-known example that has been used a lot for both linear and nonlinear problems in many different scientific fields [11,12].

A practical strategy to optimize the implementation of these series methods is to amalgamate them with integral transforms [13,14]. Transform operators often change derivatives into algebraic expressions, which cuts down on the number of calculations needed to find the solution terms [13,14]. Alzaki and Jassim presented the fractional Sumudu technique for homotopy perturbation, illustrating that transform-homotopy integration is straightforward to apply and yields accurate approximations for time-fractional partial differential equations [15–17]. This reasoning supports the dominant view that, when properly employed, the transform mainly serves as a computational facilitator, with the accuracy of approximation determined by the iterative framework itself [18,19].

Yasser and Jassim recently unveiled the Yasser–Jassim (YJ) transform, an innovative Laplace-class transform, and detailed its fundamental characteristics and uses for differential and Volterra-type integral equations [20,21]. Their results show that the transformation can change the original problem into an algebraic form in the transform domain, which then makes it possible to find the solution in the original variables by inverting it [20]. Also, the body of literature on Laplace-type transforms has grown steadily, with the introduction of several related transforms, such as Elzaki, Aboodh, Mohand, and Sawi, to make it easier to simplify differential and integral models in a variety of situations [14,20].

On the other hand, iterative series methods that use power-series constructions have been shown to work very well [18,19]. The Hussein–Jassim framework was presented as an effective iterative technique for nonlinear fractional equations, along with a detailed examination of convergence [22,23]. Additionally, approximate analyses of fractional systems employing non-singular operators have been investigated, as illustrated by recent studies on fractional operators devoid of singular kernels and their associated applications, which further validate the effectiveness of recursive series formulations for coupled models [8–10].

Motivated by these developments, this paper amalgamates the Yasser–Jassim transforms with two separate iterative methodologies: (i) a transform-assisted HPM formulation and (ii) a transform-assisted Hussein–Jassim iterative construction. The goal is not to replace the basic iterative methods but to make them easier to use and less reliant on algebra, while keeping the same convergence properties [11,15–17,20,22,23]. This is in line with earlier work on transform-assisted decomposition and iterative schemes for both fractional and whole number models [8,18–21,24].

2-Preliminaries

This section depends on the type of study. In this section, authors should identify previous works related to the topic to show the scientific gap in the presented research.

Definition 2.1 [2,3,5] if $\wp(\ell) \in C([\mathfrak{a},\mathfrak{b}]), \propto > 0, and\ \mathfrak{a} < \ell < \mathfrak{b}$, then the Riemann-Liouville fractional integral of order $\propto$ defined as follows:

$$I_\ell^\propto \wp(\ell) = \frac{1}{\Gamma(\propto)} \int_0^\ell \frac{\wp(\tau)}{(\ell-\tau)^{1-\propto}} d\tau$$

Where Γ is the well-known Gamma function

The properties of the Riemann-Liouville fractional integral are as follows:

I. $I_\ell^\propto I_\ell^\beta \wp(\ell) = I_\ell^{\propto+\beta} \wp(\ell)$

II. $I_\ell^\propto I_\ell^\beta \wp(\ell) = I_\ell^\beta I_\ell^\propto \wp(\ell)$

III. $I_\ell^\propto \ell^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\propto+\beta+1)} \ell^{\propto+\beta}$

where $\propto$ and β are greater than zero and β is a real number

Definition 2.2 [2–4] Caputo fractional derivative

Let $\wp(\ell) \in H^1(\mathfrak{a},\mathfrak{b}), \mathfrak{a} < \mathfrak{b}$, the Caputo fractional derivative (CDF) of order α defined as follows:

$${}_a^cD_\ell^\propto \wp(\ell) = \frac{1}{\Gamma(\mathfrak{n}-\propto)} \int_a^\ell (\ell-\tau)^{\mathfrak{n}-\propto-1} \wp^{(\mathfrak{n})}(\tau) d\tau$$

Where $\mathfrak{n}-1 < \propto < \mathfrak{n}, \mathfrak{n} \in N$.

The following are the basic properties of the ${}_{\mathfrak{a}}^{c}D_{\ell}^{\propto}$:

I. ${}_{a}^{c}D_{\ell}^{\propto}\lambda = 0$

II. ${}_{a}^{c}D_{\ell}^{\propto}I^{\propto}\wp(\ell) = \wp(\ell)$

III. ${}_{a}^{c}D_{\ell}^{\propto}n^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\propto+1)}n^{\beta-\propto}$

IV. ${}_{a}^{c}D_{\ell}^{\propto}\,{}_{a}^{c}D_{\ell}^{\beta}\wp(\ell) = {}_{a}^{c}D_{\ell}^{\propto+\beta}\wp(\ell) = {}_{a}^{c}D_{\ell}^{\beta}\,{}_{a}^{c}D_{\ell}^{\propto}\wp(\ell)$

V. ${}_{a}^{c}D_{\ell}^{\propto}\big(\lambda\wp(\ell) + \sigma g(\ell)\big) = \lambda\,{}_{\mathfrak{a}}^{c}D_{\ell}^{\propto}\big(\wp(\ell)\big)+\sigma\,{}_{\mathfrak{a}}^{c}D_{\ell}^{\propto}\big(g(\ell)\big)$

VI. $I^{\propto}\,{}_{a}^{c}D_{\ell}^{\propto}\wp(\ell) = \wp(\ell) - \sum_{k=0}^{\mathfrak{n}-1}\wp^{k}(0)\frac{\ell^{k}}{k!}$

where λ and σ are constants.

Definition 2.3 [20]. Let $\wp(\ell)$ be an integral function defined for $\ell \geqslant 0$, we define a Yasser Jassim transform of $\wp(\ell)$ by the formula.

$$\mathcal{H}\{\wp(\ell)\} = K(\mathfrak{a}) = \mathfrak{a}\int_{0}^{\infty} e^{-\frac{1}{\sqrt{\mathfrak{a}}}\ell}\wp(\ell)d\ell \qquad , \ where\ \mathfrak{a} \neq 0$$

And $\mathfrak{a}$ is Yasser Jassim transform parameters.

The inverse Yasser Jassim transform is used to obtain the original function $\wp(\ell)$.

$$\wp(\ell) = \mathcal{H}^{-1}\{K(\mathfrak{a})\}.$$

The Yasser Jassim transform possesses the following significant characteristics:

(i) For functions $\wp_1(\ell)$ and $\wp_2(\ell)$ Yasser Jassim transformations and constants that are defined $c_1, c_2 \in R$, then

$$\mathcal{H}[c_1\wp_1(\ell) + c_2\wp_2(\ell)] = c_1\mathcal{H}[\wp_1(\ell)] + c_2\mathrm{H}[\wp_2(\ell)].$$

(ii) The Yasser Jassim transform of the 1^{th} derivative of the function $\wp(\ell)$ is

$$\mathcal{H}[\wp'(\ell)] = \frac{1}{\sqrt{\mathfrak{a}}}\mathcal{H}[\wp(\ell)] - \mathfrak{a}\wp(0).$$

(iii) The Yasser Jassim transform of the 2^{th} derivative of the function $\wp(\ell)$ is

$$\mathcal{H}[\wp''(\ell)] = \frac{1}{\mathfrak{a}}\mathcal{H}[\wp(\ell)] - \sqrt{\mathfrak{a}}\,\wp(0) - \mathfrak{a}\wp'(0).$$

(iv) The Yasser Jassim transform of the function $\wp(\ell)$ for the fractional derivative of order $\propto$, is

$$\mathcal{H}\left[\wp^{(\propto)}(\ell)\right] = \frac{1}{\sqrt{\mathfrak{a}}^{\propto}}\mathcal{H}[\wp(\ell)] - \sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{a}}{\sqrt{\mathfrak{a}}^{\propto-k-1}}\ \wp^{(k)}(0)\ ,\ \ \mathfrak{n}-1 <\propto< \mathfrak{n}$$

gives some helpful details about the Yasser Jassim transformations of a few fundamental functions.

$\mathcal{H}[1] = \mathfrak{a}\sqrt{\mathfrak{a}}$.

$\mathcal{H}\left[e^{\mathfrak{b}\ell}\right] = \frac{\mathfrak{a}\sqrt{\mathfrak{a}}}{1-\mathfrak{b}\sqrt{\mathfrak{a}}}$.

$\mathcal{H}[\ell^{\mathfrak{n}}] = \Gamma(\propto +1)\mathfrak{a}\sqrt{\mathfrak{a}}^{\propto+1}$.

$\mathcal{H}[sin(\mathfrak{b}\ell)] = \frac{\mathfrak{a}^2\mathfrak{b}}{1+\mathfrak{a}\mathfrak{b}^2}$.

$\mathcal{H}[cos(\mathfrak{b}\ell)] = \frac{\mathfrak{a}\sqrt{\mathfrak{a}}}{1+\mathfrak{a}\mathfrak{b}^2}$.

$\mathcal{H}[sinh(\mathfrak{b}\ell)] = \frac{\mathfrak{a}^2\mathfrak{b}}{1-\mathfrak{a}\mathfrak{b}^2}$.

$\mathcal{H}[cosh(\mathfrak{b}\ell)] = \frac{\mathfrak{a}\sqrt{\mathfrak{a}}}{1-\mathfrak{a}\mathfrak{b}^2}$.

3- Methodology of the Proposed Methods

3.1 Analysis of Fractional Yasser Jassim Homotopy Perturbation Method

Examine the general nonlinear differential equation that follows:

$$ {}^{c}D_{\ell}^{\propto}\wp(n,\ell) + L(\wp(n,\ell)) + N(\wp(n,\ell)) = g(n,\ell), \qquad \mathfrak{n}-1 <\propto< \mathfrak{n} \tag{1}$$

using the initial condition.

$$\wp_{\ell}^{(k)}(n,0) = \lambda_k(n), \ k = 0,1,2 \dots, \mathfrak{n}-1.$$

where $\lambda_k(n)$ are provided sufficiently functions. And $\wp(n,\ell)$ represents the analytic function for $\ell \geq 0$, ${}^{c}D_{\ell}^{\propto}$ is Caputo operator, L and N denote the linear and nonlinear operators, respectively, $g(n,\ell)$ is a given function. The Yasser Jassim transform on both sides of (1), we arrive at:

$$\mathcal{H}\{{}^{c}D_{\ell}^{\propto}\wp(n,\ell)\} = \mathcal{H}\{g(n,\ell) - L(\wp(n,\ell)) - N(\wp(n,\ell))\}, \tag{2}$$

We obtain by utilizing the Yasser Jassim transform's property,

$$\frac{\mathcal{H}\{\wp(n,\ell)\}}{\sqrt{\mathfrak{a}}^{\propto}} = \sum_{k=0}^{\mathfrak{n}-1} \frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{\propto-k-1}} + \mathcal{H}\{g(n,\ell) - L(\wp(n,\ell)) - N(\wp(n,\ell))\},$$

$$\mathcal{H}\{\wp(n,\ell)\} = \sum_{k=0}^{\mathfrak{n}-1} \frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}} + \sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{g(n,\ell) - L(\wp(n,\ell)) - N(\wp(n,\ell))\}, \tag{3}$$

In the second stage, we apply the inverse Yasser Jassim transform to both sides of (3) to obtain,

$$\wp(n,\ell) = \mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1} \frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{g(n,\ell) - L(\wp(n,\ell)) - N(\wp(n,\ell))\}\right\}, \tag{4}$$

We implement the HPM:

$$\wp(n,\ell) = \sum_{\mathrm{i}=0}^{\infty} p^{\mathrm{i}}\wp_{\mathrm{i}}, \tag{5}$$

Thus, the nonlinear term decomposed as follows:

$$N(\wp(n,\ell)) = \sum_{\mathrm{i}=0}^{\infty} p^{\mathrm{i}} A_{\mathrm{i}}, \tag{6}$$

Where:

$$A_{\mathrm{i}} = \frac{1}{\mathrm{i}!}\frac{\partial}{\partial p^{\mathrm{i}}}\left[N\left[\sum_{\mathrm{i}=0}^{\infty} p^{\mathrm{i}}\wp_{\mathrm{i}}\right]\right], \tag{7}$$

When we substitute (5) and (6) into (4), we obtain:

$$\sum_{\mathrm{i}=0}^{\infty} p^{\mathrm{i}}\wp_{\mathrm{i}} = \mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1} \frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{g(n,\ell)\}\right\} - p\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{\begin{matrix} L\left(\sum_{\mathrm{i}=0}^{\infty} p^{\mathrm{i}}\wp_{\mathrm{i}}\right) \\ +N\left(\sum_{\mathrm{i}=0}^{\infty} p^{\mathrm{i}}\wp_{\mathrm{i}}\right)\end{matrix}\right\}\right\}, \tag{8}$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$p^0: \wp_0(n,\ell) = \mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1} \frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{g(n,\ell)\}\right\}$$

$$p^1: \wp_1(n,\ell) = -\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\wp_0 + A_0\}\right\}$$

$$p^2: \wp_2(n,\ell) = -\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\wp_1 + A_1\}\right\}$$

$$p^{\mathfrak{n}}: \wp_{\mathfrak{n}}(n,\ell) = -\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\wp_{\mathfrak{n}-1} + A_{\mathfrak{n}-1}\}\right\}$$

$$\vdots$$

Consequently, we use truncated series to estimate the analytical result $\wp(n,\ell)$

$$\wp(n,\ell)=\lim_{p\to 1}\sum_{\mathrm{i}=0}^{\infty}p^{\mathrm{i}}\wp_{\mathrm{i}}$$

3.2 Analysis of Fractional Yasser Jassim Hussein Jassim Method

Examine the general nonlinear differential equation that follows:

$$^{c}D_{\ell}^{\propto}\wp(n,\ell)+L\big(\wp(n,\ell)\big)+N\big(\wp(n,\ell)\big)=g(n,\ell),\qquad \mathfrak{n}-1<\propto<\mathfrak{n} \tag{9}$$

using the initial condition.

$$\wp_{\ell}^{(k)}(n,0)=\lambda_k(n),\ k=0,1,2\ldots,\mathfrak{n}-1.$$

where $\lambda_k(n)$ are provided sufficiently functions. And $\wp(n,\ell)$ represents the analytic function for $\ell\geq 0$, $^{c}D_{\ell}^{\propto}$ is caputo operator, L and N denote the linear and nonlinear operators, respectively, $g(n,\ell)$ is a given function. By applying the Yasser Jassim transform to (9), we arrive at:

$$\mathcal{H}\{^{c}D_{\ell}^{\propto}\wp(n,\ell)\}=\mathcal{H}\big\{g(n,\ell)-L\big(\wp(n,\ell)\big)-N\big(\wp(n,\ell)\big)\big\}, \tag{10}$$

We obtain by utilizing the Yasser Jassim transform's property,

$$\mathcal{H}\{\wp(n,\ell)\}=\sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}+\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\big\{g(n,\ell)-L\big(\wp(n,\ell)\big)-N\big(\wp(n,\ell)\big)\big\}, \tag{11}$$

Operating with Yasser Jassim inverse on both sides of (11), we obtain

$$\wp(n,\ell)=\mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\}+\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\big\{g(n,\ell)-L\big(\wp(n,\ell)\big)-N\big(\wp(n,\ell)\big)\big\}\right\}, \tag{12}$$

We now use Maclaurin's expansion with respect to ℓ to rewrite (12).
We formulate (12) using Maclaurin's extension with respect to ℓ.

$$\wp(n,\ell)=\sum_{\mathrm{i}=0}^{\infty}\frac{\ell^{\mathrm{i}\propto}}{\Gamma(\mathrm{i}\propto+1)}\,{}^{c}D_{\ell}^{\mathrm{i}\propto}\left(\mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\}+\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{\begin{matrix}g(n,\ell)-\\ L\big(\wp(n,\ell)\big)-N\big(\wp(n,\ell)\big)\end{matrix}\right\}\right\}\right)_{\ell=0}, \tag{13}$$

Where $^{c}D_{\ell}^{\mathrm{i}\propto}$ denotes the Caputo fractional derivative of order $\mathrm{i}\propto$.
Express the solution in series form.

$$\wp(n,\ell)=\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}}, \tag{14}$$

Substituting (14) into (13) yields the result that:

$$\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}}=\sum_{\mathrm{i}=0}^{\infty}\frac{\ell^{\mathrm{i}\propto}}{\Gamma(\mathrm{i}\propto+1)}\,{}^{c}D_{\ell}^{\mathrm{i}\propto}\left(\mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\}+\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{g(n,\ell)-L(\textstyle\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}})-N(\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}})\}\right\}\right)_{\ell=0} \tag{15}$$

Substituting $\mathrm{i}=\mathrm{i}+1$, rewrite (15) as.

$$\wp_0+\sum_{\mathrm{i}=1}^{\infty}\wp_{\mathrm{i}}=\sum_{\mathrm{i}=0}^{\infty}\frac{\ell^{\mathrm{i}\propto}}{\Gamma(\mathrm{i}\propto+1)}\,{}^{c}D_{\ell}^{\mathrm{i}\propto}\left(\begin{matrix}\mathcal{H}^{-1}\left\{\sum_{k=0}^{\mathfrak{n}-1}\frac{\mathfrak{a}\lambda_k(n)}{\sqrt{\mathfrak{a}}^{-(k+1)}}\right\}+\\ \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{g(n,\ell)-L(\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}})-N(\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}})\}\right\}\end{matrix}\right)_{\ell=0}, \tag{16}$$

by comparing.

$$\begin{cases} \wp_0 = \wp(n,0) = \lambda_0(n), \\ \wp_1 = \frac{\ell^{\alpha}}{\Gamma(\alpha+1)} {}^{c}D_{\ell}^{\alpha}\left(\mathcal{H}^{-1}\left\{\sum_{k=0}^{n-1}\frac{a\lambda_k(n)}{\sqrt{a}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{g(n,\ell) - L(\wp_0) - N(\wp_0)\}\right\}\right), \\ \wp_2 = \frac{\ell^{2\alpha}}{\Gamma(2\alpha+1)} {}^{c}D_{\ell}^{2\alpha}\left(\mathcal{H}^{-1}\left\{\sum_{k=0}^{n-1}\frac{a\lambda_k(n)}{\sqrt{a}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{g(n,\ell) - L(\wp_0+\wp_1) - N(\wp_0+\wp_1)\}\right\}\right)_{\ell=0}, \\ \wp_3 = \frac{\ell^{3\alpha}}{\Gamma(3\alpha+1)} {}^{c}D_{\ell}^{3\alpha}\left(\mathcal{H}^{-1}\left\{\sum_{k=0}^{n-1}\frac{a\lambda_k(n)}{\sqrt{a}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{g(n,\ell) - L(\wp_0+\wp_1+\wp_2) - N(\wp_0+\wp_1+\wp_2)\}\right\}\right), \\ \vdots \\ \wp_{i+1} = \frac{\ell^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} {}^{c}D_{\ell}^{(i+1)\alpha}\left(\mathcal{H}^{-1}\left\{\sum_{k=0}^{n-1}\frac{a\lambda_k(n)}{\sqrt{a}^{-(k+1)}}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{g(n,\ell) - L(\sum_{j=0}^{i}\wp_j) - N(\sum_{j=0}^{i}\wp_j)\}\right\}\right)_{\ell=0}, \end{cases}$$

The approximate solution is.

$$\wp(n,\ell) = \wp_0 + \wp_1 + \wp_2 + \cdots = \sum_{i=0}^{\infty}\wp_i.$$

4-Illustrative examples

This section will put into effect the proposed method for solving time fractional partial differential equations.

4.1 Applications on Fractional Yasser Jassim Homotopy Perturbation Method

Example. 4.1.1 Consider the following time–fractional partial differential equation of Caputo type subject to the prescribed initial conditions.

$$ {}^{c}D_{\ell}^{2\alpha}\wp(n,\ell) - \wp_{nn}^2(n,\ell) + \wp^2(n,\ell) = 0, \qquad 0 < \alpha \le 1 \tag{17}$$

with initial conditions, $\wp(n,0) = 0, \wp_{\ell}(n,0) = \cos(n)$.

The Yasser Jassim transform on both sides of (17), we obtain:

$$\mathcal{H}\{{}^{c}D_{\ell}^{2\alpha}\wp(n,\ell)\} = \mathcal{H}\{\wp_{nn}^2(n,\ell) - \wp^2(n,\ell)\},$$

$$\mathcal{H}\{\wp(n,\ell)\} - a\sqrt{a}^{\alpha+1}\cos(n) = \sqrt{a}^{\alpha}\mathcal{H}\{\wp_{nn}^2(n,\ell) - \wp^2(n,\ell)\}, \tag{18}$$

Operating with Yasser Jassim inverse on both sides of (18), we obtain:

$$\wp(n,\ell) = \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\cos(n) + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{\wp_{nn}^2(n,\ell) - \wp^2(n,\ell)\}\right\}, \tag{19}$$

According to the HPM, and substituting.

$$\wp(n,\ell) = \sum_{i=0}^{\infty} p^i\,\wp_i, \qquad \wp_{nn}^2(n,\ell) = \sum_{i=0}^{\infty} p^i A_i, \qquad \wp^2(n,\ell) = \sum_{i=0}^{\infty} p^i B_i, \tag{20}$$

by Substituting (20) in (19), we get:

$$\wp(n,\ell) = \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\cos(n) + p\mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{\sum_{i=0}^{\infty} p^i A_i - \sum_{i=0}^{\infty} p^i B_i\right\}\right\}, \tag{21}$$

By comparing.

$$p^0: \ \wp_0 = \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\cos(n)$$

$$p^1: \ \wp_1 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{A_0 - B_0\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{\wp_{0nn}^2 - \wp_0^2\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{\frac{\ell^{2\alpha}}{\Gamma(2\alpha+1)}\cos(n) - \frac{\ell^{2\alpha}}{\Gamma(2\alpha+1)}\cos(n)\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{0\}\right\} = 0$$

$$\vdots$$

The approximate solution is

$$\wp(n,\ell) = \sum_{i=0}^{\infty} \wp_i = \wp_0 + \wp_1 + \wp_2 + \wp_3 + \cdots$$

$$\wp_a(n,\ell) = \frac{\ell^{\propto}}{\Gamma(\propto+1)}\cos(n),$$

Which is the exact solution

$$\wp_e(n,\ell) = \frac{\ell^{\propto}}{\Gamma(\propto+1)}\cos(n).$$

Table 1. Approximate solution, exact solution, and absolute errors for (17) with different values of α.

ℓ, n	$\wp_{\propto=0.6}$	$\wp_{\propto=0.7}$	$\wp_{\propto=0.8}$	$\wp_{\propto=0.9}$	$\wp_{\propto=1}$	$\wp_e$	$\lvert\wp_{\propto=1} - \wp_e\rvert$
0.050	0.185241	0.135003	0.097612	0.070058	0.049938	0.049938	0.000000
0.100	0.279720	0.218491	0.169315	0.130243	0.099500	0.099500	0.000000
0.150	0.354526	0.288382	0.232723	0.186427	0.148316	0.148316	0.000000
0.200	0.417611	0.349611	0.290369	0.239394	0.196013	0.196013	0.000000
0.250	0.472005	0.404065	0.343169	0.289308	0.242228	0.242228	0.000000
0.300	0.519191	0.452637	0.391494	0.336121	0.286601	0.286601	0.000000
0.350	0.559985	0.495786	0.435475	0.379691	0.328780	0.328780	0.000000
0.400	0.594870	0.533752	0.475125	0.419830	0.368424	0.368424	0.000000
0.450	0.624142	0.566651	0.510387	0.456331	0.405201	0.405201	0.000000
0.500	0.647990	0.594533	0.541172	0.488981	0.438791	0.438791	0.000000
0.550	0.666534	0.617404	0.567373	0.517564	0.468888	0.468888	0.000000
0.600	0.679860	0.635251	0.588875	0.541873	0.495201	0.495201	0.000000
0.650	0.688026	0.648048	0.605565	0.561709	0.517454	0.517454	0.000000
0.700	0.691081	0.655766	0.617336	0.576887	0.535390	0.535390	0.000000
0.750	0.689067	0.658382	0.624089	0.587235	0.548767	0.548767	0.000000
0.800	0.682028	0.655876	0.625739	0.592600	0.557365	0.557365	0.000000
0.850	0.670011	0.648239	0.622213	0.592844	0.560986	0.560986	0.000000
0.900	0.653073	0.635472	0.613456	0.587850	0.559449	0.559449	0.000000
0.950	0.631275	0.617592	0.599427	0.577522	0.552599	0.552599	0.000000
1.000	0.604693	0.594628	0.580107	0.561782	0.540302	0.540302	0.000000

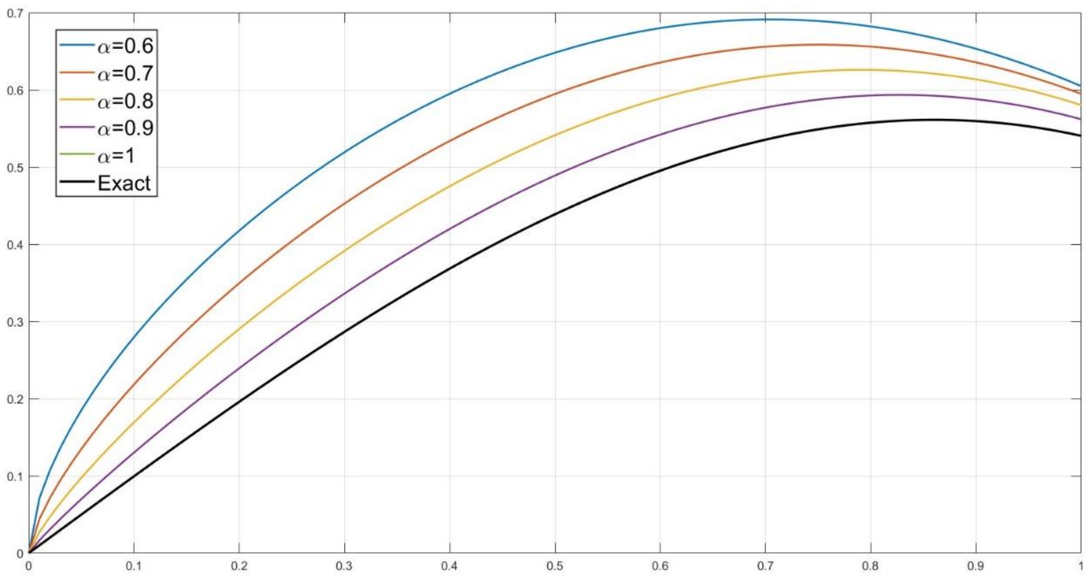

Figure 1. Approximate solution and exact solution for (17) with different values of α.

Example. 4.1.2 Consider the following time–fractional Burgers' equation of Caputo type.

$$ {}^{c}D_{\ell}^{\propto}\wp(n,\ell)-\wp_{nn}(n,\ell)+\wp(n,\ell)\wp_{n}(n,\ell)=0, \qquad 0<\propto\leq 1 \tag{22}$$

with initial conditions, $\wp(n,0)=n$.

The Yasser Jassim transform on both sides of (22), we obtain

$$\mathcal{H}\{\wp(n,\ell)\}=n+\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\wp_{nn}(n,\ell)-\wp(n,\ell)\wp_{n}(n,\ell)\}, \tag{23}$$

Operating with Yasser Jassim inverse on both sides of (23), we obtain

$$\wp(n,\ell)=n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\wp_{nn}(n,\ell)-\wp(n,\ell)\wp_{n}(n,\ell)\}\right\}, \tag{24}$$

According to the HPM, and substituting,

$$\wp(n,\ell)=\sum_{\mathrm{i}=0}^{\infty}p^{\mathrm{i}}\,\wp_{\mathrm{i}}, \qquad \wp(n,\ell)\wp_{n}(n,\ell)=\sum_{\mathrm{i}=0}^{\infty}p^{\mathrm{i}}\,A_{\mathrm{i}},$$

In (24), we have.

$$\sum_{\mathrm{i}=0}^{\infty}p^{\mathrm{i}}\,\wp_{\mathrm{i}}=n+p\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{\sum_{\mathrm{i}=0}^{\infty}p^{\mathrm{i}}\,\wp_{\mathrm{i}nn}-\sum_{\mathrm{i}=0}^{\infty}p^{\mathrm{i}}\,A_{\mathrm{i}}\right\}\right\}, \tag{25}$$

We get the following set of equations by comparing the terms with comparable powers of p:

$$p^{0}:\ \wp_{0}=n$$

$$p^{1}:\ \wp_{1}=\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\wp_{0nn}-A_{0}\}\right\},$$

$$=\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{(n)_{nn}-\wp_{0}\wp_{0n}\}\right\}=\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{(n)_{nn}-(n)(1)\}\right\},$$

$$=\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{-n\}\right\}=\mathcal{H}^{-1}\left\{-n\mathrm{a}\sqrt{\mathrm{a}}^{\propto+1}\right\}=-n\frac{\ell^{\propto}}{\Gamma(\propto+1)}$$

$$p^{2}:\ \wp_{2}=\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\wp_{1nn}-A_{1}\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{\left(n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{nn} - (\wp_1\wp_{0n} + \wp_0\wp_{1n})\right\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{0 - \left(-2n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\frac{2n}{\Gamma(\propto+1)}\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\ell^{\propto}\}\right\} = \mathcal{H}^{-1}\left\{2n\mathfrak{a}\sqrt{\mathfrak{a}}^{2\propto+1}\right\} = 2n\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}$$

$$p^3:\ \wp_3 = \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\wp_{2nn} - A_2\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{\left(n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{nn} - (\wp_2\wp_{0n} + \wp_0\wp_{2n} + \wp_1\wp_{1n})\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{0 - \left(4n\frac{\ell^{2\propto}}{\Gamma(2\propto+1)} + n\frac{\ell^{2\propto}}{\Gamma(\propto+1)^2}\right)\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{-n\left(4 + \frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2}\right)\mathfrak{a}\sqrt{\mathfrak{a}}^{3\propto+1}\right\} = -n\left(4 + \frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2}\right)\frac{\ell^{3\propto}}{\Gamma(3\propto+1)}$$

$$\vdots$$

the approximate solution is

$$\wp(n,\ell) = n\left(1 - \frac{\ell^{\propto}}{\Gamma(\propto+1)} + \frac{2\ell^{2\propto}}{\Gamma(2\propto+1)} - \left(4 + \frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2}\right)\frac{\ell^{3\propto}}{\Gamma(3\propto+1)} + \cdots\right)$$

The exact solution when $\propto = 1$ is

$$\wp_e(n,\ell) = \frac{n}{1+\ell}.$$

Table 2. Approximate solution, exact solution, and absolute errors for (22) with different values of α.

ℓ, n	$\wp_{\propto=0.6}$	$\wp_{\propto=0.7}$	$\wp_{\propto=0.8}$	$\wp_{\propto=0.9}$	$\wp_{\propto=1}$	$\wp_e$	$\lvert\wp_{\propto=1} - \wp_e\rvert$
0.050	0.042489	0.044224	0.045621	0.046743	0.047619	0.047619	0.000000
0.100	0.078255	0.082461	0.085743	0.088523	0.090900	0.090909	0.000009
0.150	0.108335	0.116220	0.121786	0.126356	0.130369	0.130435	0.000066
0.200	0.131982	0.145428	0.154088	0.160701	0.166400	0.166667	0.000267
0.250	0.148029	0.169469	0.182512	0.191697	0.199219	0.200000	0.000781
0.300	0.155132	0.187427	0.206599	0.219229	0.228900	0.230769	0.001869
0.350	0.151869	0.198189	0.225654	0.242975	0.255369	0.259259	0.003891
0.400	0.136768	0.200496	0.238788	0.262416	0.278400	0.285714	0.007314
0.450	0.108339	0.192982	0.244949	0.276864	0.297619	0.310345	0.012726
0.500	0.065078	0.174193	0.242938	0.285463	0.312500	0.333333	0.020833
0.550	0.005479	0.142599	0.231432	0.287207	0.322369	0.354839	0.032470
0.600	-0.071965	0.096613	0.208989	0.280940	0.326400	0.375000	0.048600
0.650	-0.168758	0.034590	0.174061	0.265367	0.323619	0.393939	0.070321
0.700	-0.286397	-0.045158	0.125003	0.239058	0.312900	0.411765	0.098865
0.750	-0.426376	-0.144362	0.060077	0.200451	0.292969	0.428571	0.135603
0.800	-0.590179	-0.264790	-0.022544	0.147856	0.262400	0.444444	0.182044
0.850	-0.779285	-0.408245	-0.124766	0.079462	0.219619	0.459459	0.239841
0.900	-0.995165	-0.576557	-0.248573	-0.006665	0.162900	0.473684	0.310784
0.950	-1.239282	-0.771589	-0.396023	-0.112576	0.090369	0.487179	0.396811
1.000	-1.513091	-0.995225	-0.569247	-0.240438	0.000000	0.500000	0.500000

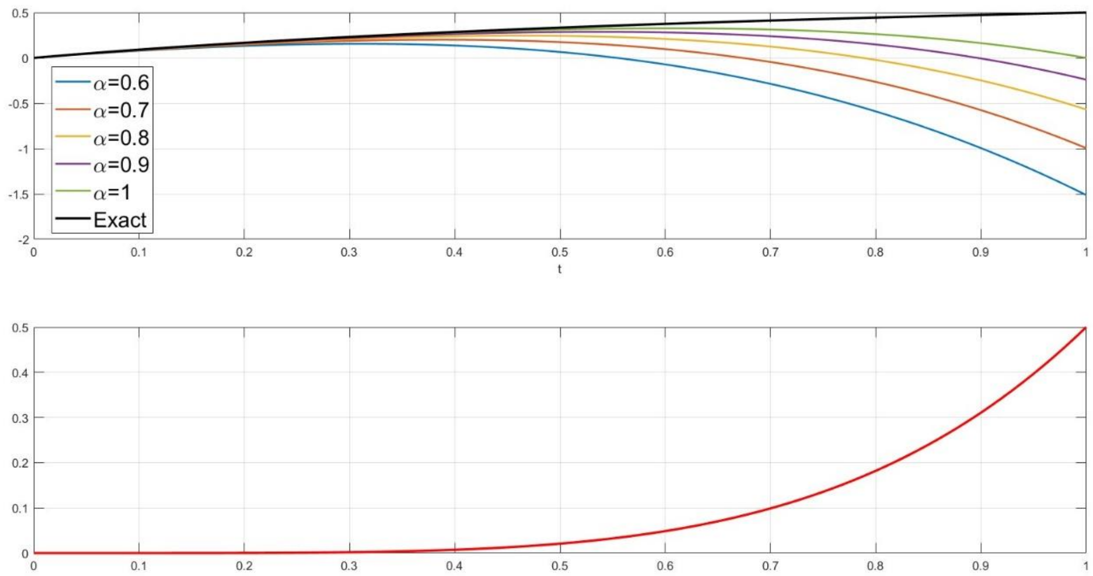

Figure 2. Approximate solution, exact solution, and absolute errors for (22) with different values of α

Example. 4.1.3 Consider the following time–fractional nonlinear dispersive partial differential equation of Caputo type.

$$ {}^{C}D_{\ell}^{\alpha}\wp(n,\ell) - 6\wp(n,\ell)\wp_{n}(n,\ell) + \wp_{nnn}(n,\ell) = 0, \qquad 0 < \alpha \leq 1 \tag{26} $$

with initial conditions, $\wp(n,0) = 6n$.

The Yasser Jassim transform and taking Yasser Jassim inverse on both sides of (26), we obtain

$$ \wp(n,\ell) = 6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6\wp(n,\ell)\wp_{n}(n,\ell) - \wp_{nnn}(n,\ell)\}\right\}, \tag{27} $$

According to the HPM, and substituting

$$ \wp(n,\ell) = \sum_{i=0}^{\infty} p^{i}\,\wp_{i}, \qquad \wp(n,\ell)\wp_{n}(n,\ell) = \sum_{i=0}^{\infty} p^{i}\,A_{i}, $$

By substituting in (27).

$$ \sum_{i=0}^{\infty} p^{i}\,\wp_{i} = 6n + p\mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{6\sum_{i=0}^{\infty} p^{i}\,A_{i} - \sum_{i=0}^{\infty} p^{i}\,\wp_{innn}\right\}\right\}, \tag{28} $$

By comparing.

$$ p^{0}: \ \wp_{0} = 6n $$

$$ p^{1}: \ \wp_{1} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6A_{0} - \wp_{0nnn}\}\right\}, $$

$$ = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6\wp_{0}\wp_{0n} - (6n)_{nnn}\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6(6^{2}n) - 0\}\right\}, $$

$$ = \mathcal{H}^{-1}\left\{6^{3}\sqrt{a}^{\alpha}\mathcal{H}\{n\}\right\} = \mathcal{H}^{-1}\left\{6^{3}na\sqrt{a}^{\alpha+1}\right\} = 6^{3}n\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} $$

$$ p^{2}: \ \wp_{2} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6A_{1} - \wp_{1nnn}\}\right\}, $$

$$ = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{6(\wp_{1}\wp_{0n} + \wp_{0}\wp_{1n}) - \left(6^{3}n\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right)_{nnn}\right\}\right\} $$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{6\left(6^4 n\frac{2\ell^{\propto}}{\Gamma(\propto+1)}\right)-(0)\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{2\frac{6^5 n}{\Gamma(\propto+1)}\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\ell^{\propto}\}\right\} = \mathcal{H}^{-1}\left\{2(6^5 n)\mathrm{a}\sqrt{\mathrm{a}}^{2\propto+1}\right\} = 6^5 n\frac{2\ell^{2\propto}}{\Gamma(2\propto+1)}$$

$$p^3:\ \wp_3 = \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{6A_2-\wp_{2nnn}\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{6(\wp_2\wp_{0n}+\wp_0\wp_{2n}+\wp_1\wp_{1n})-\left(6^5 n\frac{2\ell^{2\propto}}{\Gamma(2\propto+1)}\right)_{nnn}\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{6\left(24\frac{n6^5\ell^{2\propto}}{\Gamma(2\propto+1)}+6^6 n\frac{\ell^{2\propto}}{\Gamma(\propto+1)^2}\right)-(0)\right\}\right\},$$

$$= \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{6^6 n\left(\frac{24}{\Gamma(2\propto+1)}+\frac{6}{\Gamma(\propto+1)^2}\right)\ell^{2\propto}\right\}\right\},$$

$$= 6^6 n\left(\frac{24}{\Gamma(2\propto+1)}+\frac{6}{\Gamma(\propto+1)^2}\right)\frac{\Gamma(2\propto+1)}{\Gamma(3\propto+1)}\ell^{3\propto} = 6^6 n\left(24+6\frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2}\right)\frac{\ell^{3\propto}}{\Gamma(3\propto+1)},$$

$$\vdots$$

The approximate solution is

$$\wp(n,\ell) = \sum_{i=0}^{\infty}\wp_i = \wp_0+\wp_1+\wp_2+\wp_3+\cdots$$

$$\wp(n,\ell) = 6n + n\frac{6^3\ell^{\propto}}{\Gamma(\propto+1)} + 2n\frac{6^5\ell^{2\propto}}{\Gamma(2\propto+1)} + 6^6 n\left(24+6\frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2}\right)\frac{\ell^{3\propto}}{\Gamma(3\propto+1)}+\cdots$$

The exact solution when $\propto = 1$ is

$$\wp(n,\ell) = \frac{6n}{1-6^2\ell}.$$

Table 3. Approximate solution, exact solution, and absolute errors for (26) with different values of α.

ℓ, n	$\wp_{\propto=0.6}$	$\wp_{\propto=0.7}$	$\wp_{\propto=0.8}$	$\wp_{\propto=0.9}$	$\wp_{\propto=1}$	$\wp_e$	$\lvert\wp_{\propto=1}-\wp_e\rvert$
0.001	0.013973	0.008737	0.007101	0.006486	0.006224	0.006224	0.000000
0.002	0.044057	0.022807	0.016319	0.013936	0.012927	0.012931	0.000004
0.003	0.092890	0.043021	0.027904	0.022444	0.020158	0.020179	0.000022
0.004	0.162973	0.070245	0.042154	0.032120	0.027966	0.028037	0.000072
0.005	0.256615	0.105323	0.059377	0.043082	0.036401	0.036585	0.000184
0.006	0.375981	0.149078	0.079881	0.055451	0.045516	0.045918	0.000402
0.007	0.523126	0.202317	0.103974	0.069350	0.055363	0.056150	0.000787
0.008	0.700018	0.265831	0.131968	0.084903	0.065996	0.067416	0.001419
0.009	0.908549	0.340402	0.164173	0.102239	0.077471	0.079882	0.002411
0.010	1.150551	0.426802	0.200901	0.121485	0.089843	0.093750	0.003907
0.011	1.427804	0.525794	0.242467	0.142773	0.103169	0.109272	0.006103
0.012	1.742040	0.638138	0.289186	0.166235	0.117508	0.126761	0.009252
0.013	2.094952	0.764585	0.341374	0.192007	0.132920	0.146617	0.013696
0.014	2.488195	0.905884	0.399352	0.220225	0.149466	0.169355	0.019889
0.015	2.923394	1.062779	0.463439	0.251026	0.167206	0.195652	0.028446
0.016	3.402141	1.236009	0.533958	0.284552	0.186204	0.226415	0.040211
0.017	3.926004	1.426314	0.611233	0.320944	0.206524	0.262887	0.056362
0.018	4.496524	1.634428	0.695592	0.360347	0.228231	0.306818	0.078587

0.019	5.115220	1.861084	0.787363	0.402905	0.251392	0.360759	0.109368
0.020	5.783590	2.107012	0.886876	0.448767	0.276073	0.428571	0.152498

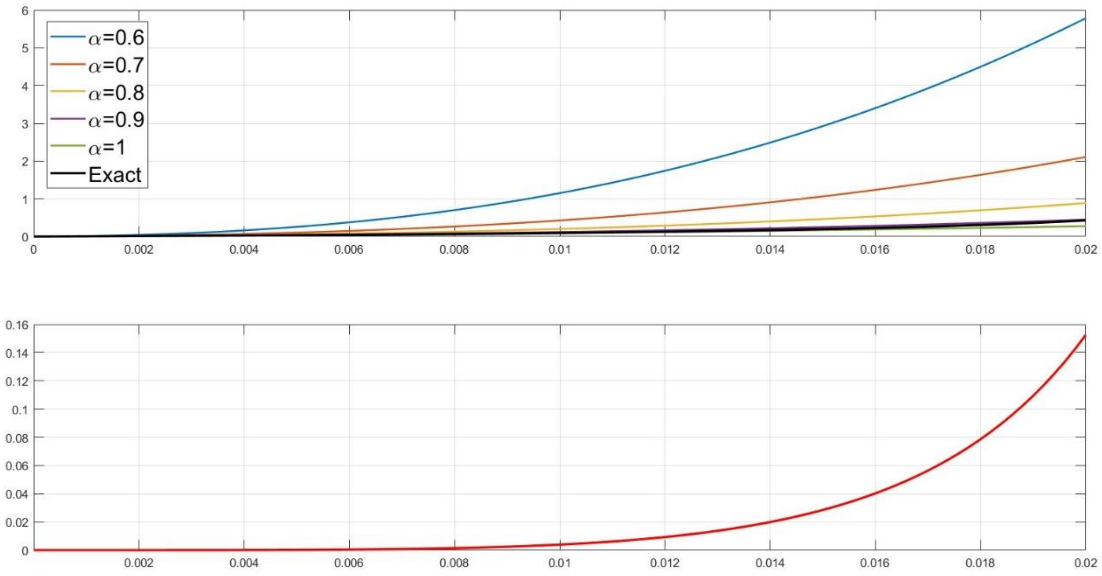

Figure 3. Approximate solution, exact solution, and absolute errors for (26) with different values of α

Example. 4.1.4 Consider the following system of coupled time–fractional partial differential equations of Caputo type.

$$\begin{aligned} {}^{c}D_{\ell}^{\alpha}\wp(\varrho,n,\ell) - \upsilon_{\varrho}(\varrho,n,\ell)\omega_{n}(\varrho,n,\ell) &= 1, \qquad 0<\alpha\leq 1 \\ {}^{c}D_{\ell}^{\alpha}\upsilon(\varrho,n,\ell) - \omega_{\varrho}(\varrho,n,\ell)\wp_{n}(\varrho,n,\ell) &= 5, \qquad 0<\alpha\leq 1 \\ {}^{c}D_{\ell}^{\alpha}\omega(\varrho,n,\ell) - \wp_{\varrho}(\varrho,n,\ell)\upsilon_{n}(\varrho,n,\ell) &= 5, \qquad 0<\alpha\leq 1 \end{aligned} \tag{29}$$

with initial conditions, $\wp(\varrho,n,0)=\varrho+2n$, $\upsilon(\varrho,n,0)=\varrho-2n$, $\omega(\varrho,n,0)=-\varrho+2n$.
The Yasser Jassim transform and taking Yasser Jassim inverse on both sides of (29), we obtain

$$\begin{aligned} \wp(\varrho,n,\ell) &= \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1+\upsilon_{\varrho}(\varrho,n,\ell)\omega_{n}(\varrho,n,\ell)\}\right\}, \\ \upsilon(\varrho,n,\ell) &= \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho-2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{5+\omega_{\varrho}(\varrho,n,\ell)\wp_{n}(\varrho,n,\ell)\}\right\}, \\ \omega(\varrho,n,\ell) &= \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{-\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{5+\wp_{\varrho}(\varrho,n,\ell)\upsilon_{n}(\varrho,n,\ell)\}\right\}, \end{aligned} \tag{30}$$

According to the HPM, and substituting

$$\begin{aligned} \wp(\varrho,n,\ell) &= \sum_{i=0}^{\infty} p^{i}\wp_{i}, &\qquad \upsilon_{\varrho}(\varrho,n,\ell)\omega_{n}(\varrho,n,\ell) &= \sum_{i=0}^{\infty} p^{i}A_{i}, \\ \upsilon(\varrho,n,\ell) &= \sum_{i=0}^{\infty} p^{i}\upsilon_{i}, &\qquad \omega_{\varrho}(\varrho,n,\ell)\wp_{n}(\varrho,n,\ell) &= \sum_{i=0}^{\infty} p^{i}B_{i}, \\ \omega(\varrho,n,\ell) &= \sum_{i=0}^{\infty} p^{i}\omega_{i}, &\qquad \wp_{\varrho}(\varrho,n,\ell)\upsilon_{n}(\varrho,n,\ell) &= \sum_{i=0}^{\infty} p^{i}C_{i}, \end{aligned}$$

by substituting in (30)

$$\sum_{i=0}^{\infty} p^{i}\wp_{i} = \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{1+\sum_{i=0}^{\infty} p^{i}A_{i}\right\}\right\},$$

$$\sum_{i=0}^{\infty} p^i \upsilon_i = \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho - 2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{5 + \sum_{i=0}^{\infty} p^i B_i\right\}\right\}, \tag{31}$$

$$\sum_{i=0}^{\infty} p^i \omega_i = \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{-\varrho + 2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{5 + \sum_{i=0}^{\infty} p^i C_i\right\}\right\},$$

By comparing

$$\wp_0 = \varrho + 2n$$
$$\upsilon_0 = \varrho - 2n$$
$$\omega_0 = -\varrho + 2n$$

$$\wp_1 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + A_0\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + 2\}\right\},$$
$$= \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{3\}\right\} = \mathcal{H}^{-1}\left\{3a\sqrt{a}^{\alpha+1}\right\} = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)},$$
$$\upsilon_1 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + B_0\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{5 - 2\}\right\},$$
$$= \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{3\}\right\} = \mathcal{H}^{-1}\left\{3a\sqrt{a}^{\alpha+1}\right\} = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)},$$
$$\omega_1 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + C_0\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{5 - 2\}\right\},$$
$$= \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{3\}\right\} = \mathcal{H}^{-1}\left\{3a\sqrt{a}^{\alpha+1}\right\} = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)},$$

$$\wp_2 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + A_1\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{0\}\right\} = 0,$$
$$\upsilon_2 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + B_1\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{0\}\right\} = 0,$$
$$\omega_2 = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{1 + C_1\}\right\} = \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{0\}\right\} = 0,$$
$$\vdots$$

The approximate solution is

$$\wp(\varrho, n, \ell) = \sum_{i=0}^{\infty} \wp_i = \wp_0 + \wp_1 + \wp_2 + \cdots$$
$$\wp_a(n, \ell) = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} + \varrho + 2n.$$
$$\upsilon(\varrho, n, \ell) = \sum_{i=0}^{\infty} \upsilon_i = \upsilon_0 + \upsilon_1 + \upsilon_2 + \cdots$$
$$\upsilon_a(n, \ell) = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} + \varrho - 2n.$$
$$\omega(\varrho, n, \ell) = \sum_{i=0}^{\infty} \omega_i = \omega_0 + \omega_1 + \omega_2 + \cdots$$
$$\omega_a(n, \ell) = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} - \varrho + 2n.$$

The exact solution is

$$\wp_e(\varrho, n, \ell) = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} + \varrho + 2n.$$
$$\upsilon_e(\varrho, n, \ell) = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} + \varrho - 2n.$$
$$\omega_e(\varrho, n, \ell) = 3\frac{\ell^{\alpha}}{\Gamma(\alpha+1)} - \varrho + 2n.$$

Table 4. Approximate solution, exact solution, and absolute errors for (29) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1} - \wp_e\rvert$
0.050	1.656418	1.505517	1.393203	1.310438	1.250000	1.250000	0.000000
0.100	2.043372	1.858764	1.710496	1.592692	1.500000	1.500000	0.000000
0.150	2.375657	2.174971	2.006099	1.865632	1.750000	1.750000	0.000000
0.200	2.678313	2.470166	2.288826	2.132789	2.000000	2.000000	0.000000

0.250	2.961448	2.751088	2.562538	2.395773	2.250000	2.250000	0.000000
0.300	3.230392	3.021395	2.829390	2.655507	2.500000	2.500000	0.000000
0.350	3.488381	3.283353	3.090743	2.912588	2.750000	2.750000	0.000000
0.400	3.737560	3.538489	3.347535	3.167433	3.000000	3.000000	0.000000
0.450	3.979442	3.787899	3.600445	3.420349	3.250000	3.250000	0.000000
0.500	4.215140	4.032399	3.849987	3.671571	3.500000	3.500000	0.000000
0.550	4.445508	4.272621	4.096562	3.921287	3.750000	3.750000	0.000000
0.600	4.671212	4.509064	4.340492	4.169645	4.000000	4.000000	0.000000
0.650	4.892789	4.742133	4.582040	4.416771	4.250000	4.250000	0.000000
0.700	5.110679	4.972164	4.821425	4.662769	4.500000	4.500000	0.000000
0.750	5.325244	5.199435	5.058831	4.907726	4.750000	4.750000	0.000000
0.800	5.536792	5.424184	5.294416	5.151719	5.000000	5.000000	0.000000
0.850	5.745584	5.646614	5.528315	5.394815	5.250000	5.250000	0.000000
0.900	5.951844	5.866901	5.760647	5.637071	5.500000	5.500000	0.000000
0.950	6.155768	6.085199	5.991516	5.878538	5.750000	5.750000	0.000000
1.000	6.357525	6.301642	6.221014	6.119262	6.000000	6.000000	0.000000

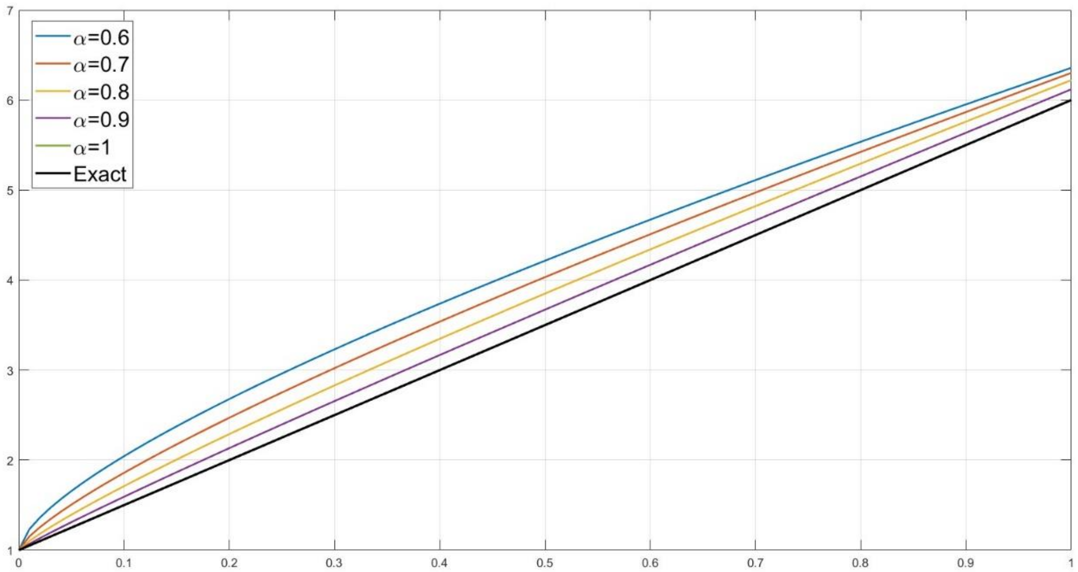

Figure 4. Approximate solution, exact solution, and absolute errors for (29) with different values of α

Table 5. Approximate solution, exact solution, and absolute errors for (29) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1}-\wp_e\rvert$
0.050	1.456418	1.305517	1.193203	1.110438	1.050000	1.050000	0.000000
0.100	1.643372	1.458764	1.310496	1.192692	1.100000	1.100000	0.000000
0.150	1.775657	1.574971	1.406099	1.265632	1.150000	1.150000	0.000000
0.200	1.878313	1.670166	1.488826	1.332789	1.200000	1.200000	0.000000
0.250	1.961448	1.751088	1.562538	1.395773	1.250000	1.250000	0.000000
0.300	2.030392	1.821395	1.629390	1.455507	1.300000	1.300000	0.000000
0.350	2.088381	1.883353	1.690743	1.512588	1.350000	1.350000	0.000000
0.400	2.137560	1.938489	1.747535	1.567433	1.400000	1.400000	0.000000
0.450	2.179442	1.987899	1.800445	1.620349	1.450000	1.450000	0.000000
0.500	2.215140	2.032399	1.849987	1.671571	1.500000	1.500000	0.000000
0.550	2.245508	2.072621	1.896562	1.721287	1.550000	1.550000	0.000000
0.600	2.271212	2.109064	1.940492	1.769645	1.600000	1.600000	0.000000
0.650	2.292789	2.142133	1.982040	1.816771	1.650000	1.650000	0.000000
0.700	2.310679	2.172164	2.021425	1.862769	1.700000	1.700000	0.000000
0.750	2.325244	2.199435	2.058831	1.907726	1.750000	1.750000	0.000000

0.800	2.336792	2.224184	2.094416	1.951719	1.800000	1.800000	0.000000
0.850	2.345584	2.246614	2.128315	1.994815	1.850000	1.850000	0.000000
0.900	2.351844	2.266901	2.160647	2.037071	1.900000	1.900000	0.000000
0.950	2.355768	2.285199	2.191516	2.078538	1.950000	1.950000	0.000000
1.000	2.357525	2.301642	2.221014	2.119262	2.000000	2.000000	0.000000

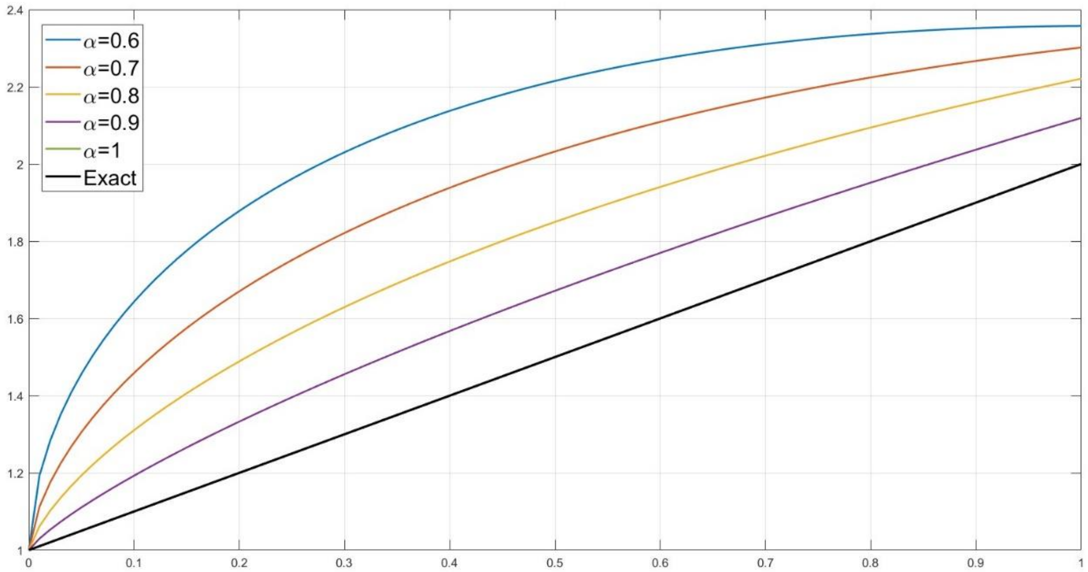

Figure 5. Approximate solution, exact solution, and absolute errors for (29) with different values of α

Table 6. Approximate solution, exact solution, and absolute errors for (29) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1}-\wp_e\rvert$
0.050	-0.343582	-0.494483	-0.606797	-0.689562	-0.750000	-0.750000	0.000000
0.100	0.043372	-0.141236	-0.289504	-0.407308	-0.500000	-0.500000	0.000000
0.150	0.375657	0.174971	0.006099	-0.134368	-0.250000	-0.250000	0.000000
0.200	0.678313	0.470166	0.288826	0.132789	0.000000	0.000000	0.000000
0.250	0.961448	0.751088	0.562538	0.395773	0.250000	0.250000	0.000000
0.300	1.230392	1.021395	0.829390	0.655507	0.500000	0.500000	0.000000
0.350	1.488381	1.283353	1.090743	0.912588	0.750000	0.750000	0.000000
0.400	1.737560	1.538489	1.347535	1.167433	1.000000	1.000000	0.000000
0.450	1.979442	1.787899	1.600445	1.420349	1.250000	1.250000	0.000000
0.500	2.215140	2.032399	1.849987	1.671571	1.500000	1.500000	0.000000
0.550	2.445508	2.272621	2.096562	1.921287	1.750000	1.750000	0.000000
0.600	2.671212	2.509064	2.340492	2.169645	2.000000	2.000000	0.000000
0.650	2.892789	2.742133	2.582040	2.416771	2.250000	2.250000	0.000000
0.700	3.110679	2.972164	2.821425	2.662769	2.500000	2.500000	0.000000
0.750	3.325244	3.199435	3.058831	2.907726	2.750000	2.750000	0.000000
0.800	3.536792	3.424184	3.294416	3.151719	3.000000	3.000000	0.000000
0.850	3.745584	3.646614	3.528315	3.394815	3.250000	3.250000	0.000000
0.900	3.951844	3.866901	3.760647	3.637071	3.500000	3.500000	0.000000
0.950	4.155768	4.085199	3.991516	3.878538	3.750000	3.750000	0.000000
1.000	4.357525	4.301642	4.221014	4.119262	4.000000	4.000000	0.000000

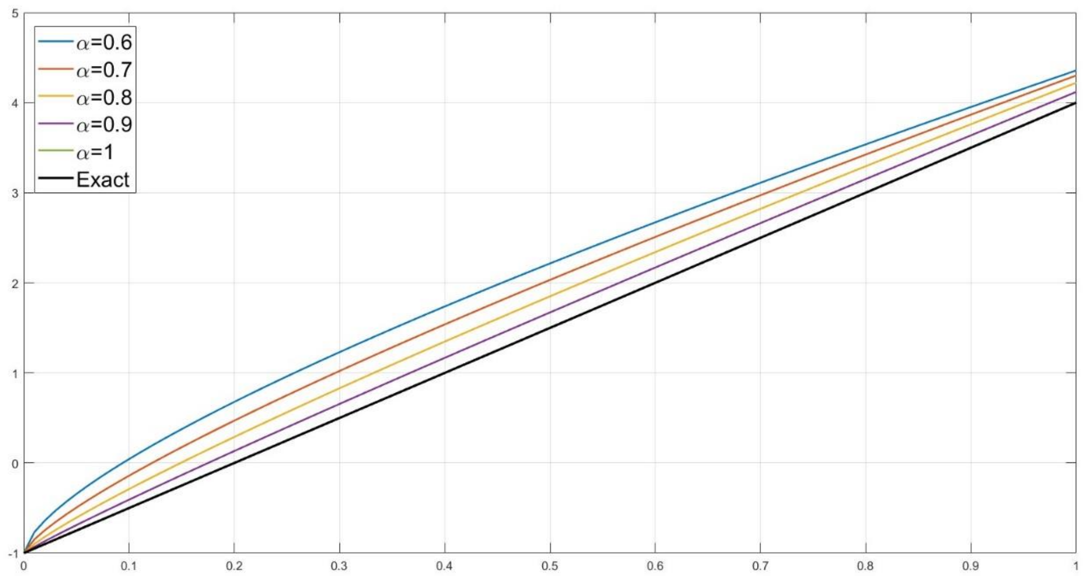

Figure 6. Approximate solution, exact solution, and absolute errors for (29) with different values of α

4.2 Applications on Fractional Yasser Jassim Hussein Jassim Method

Example. 4.2.1 Consider the following time–fractional partial differential equation of Caputo type subject to the prescribed initial conditions.

$$ {}^{c}D_{\ell}^{2\propto}\wp(n,\ell) - \wp_{nn}^{2}(n,\ell) + \wp^{2}(n,\ell) = 0, \qquad 0 < \propto \leq 1 \tag{32}$$

with initial conditions, $\wp(n,0) = 0, \wp_{\ell}(n,0) = \cos(n)$.

The Yasser Jassim transform and taking Yasser Jassim inverse on both sides of (32), we obtain

$$\wp(n,\ell) = \frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\wp_{nn}^{2}(n,\ell) - \wp^{2}(n,\ell)\}\right\}, \tag{33}$$

Using the Hussein–Jassim framework, (33) is recast into the corresponding recursive integral representation,

$$\wp(n,\ell) = \sum_{i=0}^{\infty} \frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\wp_{nn}^{2}(n,\ell) - \wp^{2}(n,\ell)\}\right\}\right)_{\ell=0}, \tag{34}$$

Assume that.

$$\wp(n,\ell) = \sum_{i=0}^{\infty} \wp_i, \qquad (\wp)_{nn}^{2} = \sum_{i=0}^{\infty} A_i, \qquad \wp^{2} = \sum_{i=0}^{\infty} B_i,$$

by Substituting in (34).

$$\sum_{i=0}^{\infty} \wp_i = \sum_{i=0}^{\infty} \frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{\sum_{i=0}^{\infty} A_i - \sum_{i=0}^{\infty} B_i\}\right\}\right)_{\ell=0}, \tag{35}$$

By comparing

$$\wp_0 = 0$$

$$\wp_1 = \frac{\ell^{\propto}}{\Gamma(\propto+1)} {}^{c}D_{\ell}^{\propto}\left(\frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{A_0 - B_0\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} {}^{c}D_{\ell}^{\propto}\left(\frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{0\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} (cos(n))_{\ell=0} = \frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n)$$

$$\wp_2 = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{A_1 - B_1\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\frac{\ell^{\propto}}{\Gamma(\propto+1)} cos(n) + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\{0\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} (0)_{\ell=0} = 0$$

$$\vdots$$

The approximate solution is

$$\wp(n,\ell) = \sum_{i=0}^{\infty} \wp_i = \wp_0 + \wp_1 + \wp_2 + \wp_3 + \cdots$$

$$\wp_\alpha(n,\ell) = \frac{\ell^{\propto}}{\Gamma(\propto+1)} \cos(n).$$

The exact solution

$$\wp_e(n,\ell) = \frac{\ell^{\propto}}{\Gamma(\propto+1)} \cos(n).$$

Table 7. Approximate solution, exact solution, and absolute errors for (32) with different values of α.

ℓ, n	$\wp_{\propto=0.6}$	$\wp_{\propto=0.7}$	$\wp_{\propto=0.8}$	$\wp_{\propto=0.9}$	$\wp_{\propto=1}$	$\wp_e$	$\lvert\wp_{\propto=1} - \wp_e\rvert$
0.050	0.185241	0.135003	0.097612	0.070058	0.049938	0.049938	0.000000
0.100	0.279720	0.218491	0.169315	0.130243	0.099500	0.099500	0.000000
0.150	0.354526	0.288382	0.232723	0.186427	0.148316	0.148316	0.000000
0.200	0.417611	0.349611	0.290369	0.239394	0.196013	0.196013	0.000000
0.250	0.472005	0.404065	0.343169	0.289308	0.242228	0.242228	0.000000
0.300	0.519191	0.452637	0.391494	0.336121	0.286601	0.286601	0.000000
0.350	0.559985	0.495786	0.435475	0.379691	0.328780	0.328780	0.000000
0.400	0.594870	0.533752	0.475125	0.419830	0.368424	0.368424	0.000000
0.450	0.624142	0.566651	0.510387	0.456331	0.405201	0.405201	0.000000
0.500	0.647990	0.594533	0.541172	0.488981	0.438791	0.438791	0.000000
0.550	0.666534	0.617404	0.567373	0.517564	0.468888	0.468888	0.000000
0.600	0.679860	0.635251	0.588875	0.541873	0.495201	0.495201	0.000000
0.650	0.688026	0.648048	0.605565	0.561709	0.517454	0.517454	0.000000
0.700	0.691081	0.655766	0.617336	0.576887	0.535390	0.535390	0.000000
0.750	0.689067	0.658382	0.624089	0.587235	0.548767	0.548767	0.000000
0.800	0.682028	0.655876	0.625739	0.592600	0.557365	0.557365	0.000000
0.850	0.670011	0.648239	0.622213	0.592844	0.560986	0.560986	0.000000
0.900	0.653073	0.635472	0.613456	0.587850	0.559449	0.559449	0.000000
0.950	0.631275	0.617592	0.599427	0.577522	0.552599	0.552599	0.000000
1.000	0.604693	0.594628	0.580107	0.561782	0.540302	0.540302	0.000000

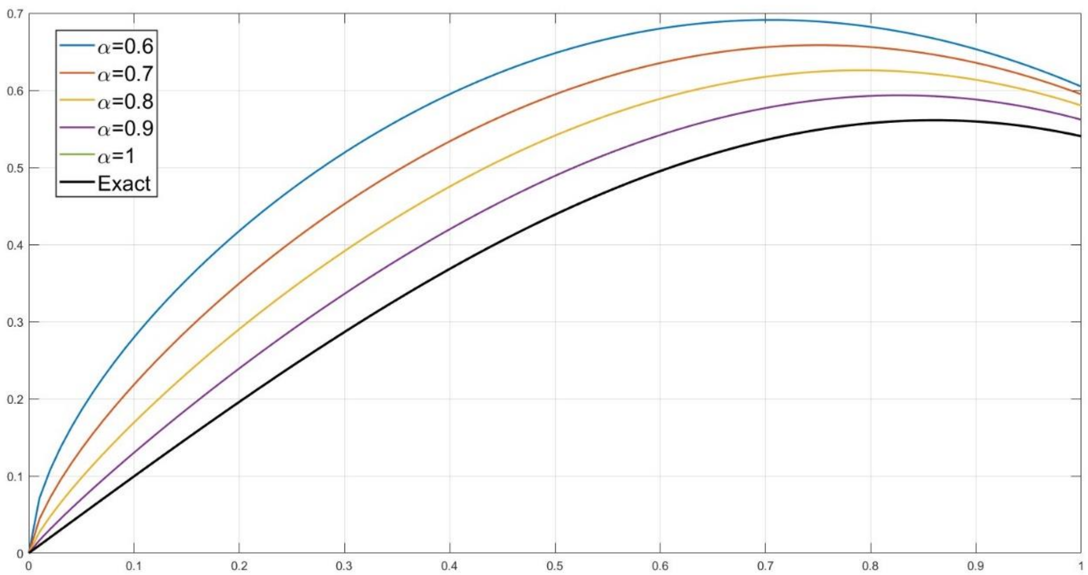

Figure 7. Approximate solution, exact solution, and absolute errors for (32) with different values of α

Example. 4.2.2 Consider the following time–fractional Burgers' equation of Caputo type.

$$ {}^{c}D_{\ell}^{\propto}\wp(n,\ell)-\wp_{nn}(n,\ell)+\wp(n,\ell)\wp_{n}(n,\ell)=0,\qquad 0<\propto\leq 1 \tag{36}$$

with initial conditions, $\wp(n,0)=n$.

The Yasser Jassim transform and taking Yasser Jassim inverse on both sides of (36), we obtain:

$$\wp(n,\ell)=n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\wp_{nn}(n,\ell)-\wp(n,\ell)\wp_{n}(n,\ell)\}\right\}, \tag{37}$$

Using the Hussein–Jassim framework, (37) is recast into the corresponding recursive integral representation,

$$\wp(n,\ell)=\sum_{\mathrm{i}=0}^{\infty}\frac{\ell^{\mathrm{i}\propto}}{\Gamma(\mathrm{i}\propto+1)}\,D_{\ell}^{\mathrm{i}\propto}\left(n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{\wp_{nn}(n,\ell)-\wp(n,\ell)\wp_{n}(n,\ell)\}\right\}\right)_{\ell=0}, \tag{38}$$

Assume that.

$$\wp(n,\ell)=\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}},$$

By substituting in (38)

$$\sum_{\mathrm{i}=0}^{\infty}\wp_{\mathrm{i}}=\sum_{\mathrm{i}=0}^{\infty}\frac{\ell^{\mathrm{i}\propto}}{\Gamma(\mathrm{i}\propto+1)}\,D_{\ell}^{\mathrm{i}\propto}\left(n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{\left(\sum_{j=0}^{\mathrm{i}-1}\wp_{j}\right)_{nn}-\left(\sum_{j=0}^{\mathrm{i}-1}\wp_{j}\right)\left(\sum_{j=0}^{\mathrm{i}-1}\wp_{j}\right)_{n}\right\}\right\}\right)_{\ell=0}. \tag{39}$$

By comparing

$$\wp_{0}=\wp(n,0)=n$$

$$\wp_{1}=\frac{\ell^{\propto}}{\Gamma(\propto+1)}\,{}^{c}D_{\ell}^{\propto}\left(n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{(\wp_{0})_{nn}-(\wp_{0})(\wp_{0})_{n}\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{\propto}}{\Gamma(\propto+1)}\,{}^{c}D_{\ell}^{\propto}\left(n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{0-n\}\right\}\right)_{\ell=0}=\frac{\ell^{\propto}}{\Gamma(\propto+1)}\,{}^{c}D_{\ell}^{\propto}\left(n-\mathcal{H}^{-1}\left\{\mathrm{a}\sqrt{\mathrm{a}}^{\propto+1}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{\propto}}{\Gamma(\propto+1)}\,{}^{c}D_{\ell}^{\propto}\left(n-n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0}=\frac{\ell^{\propto}}{\Gamma(\propto+1)}\,(n)_{\ell=0}=-n\frac{\ell^{\propto}}{\Gamma(\propto+1)},$$

$$\wp_{2}=\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{(\wp_{0}+\wp_{1})_{nn}-(\wp_{0}+\wp_{1})(\wp_{0}+\wp_{1})_{n}\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(n+\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{\left(n-n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{nn}-\left(n-n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)\left(n-n\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{n}\right\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(n-\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{n-2n\frac{\ell^{\propto}}{\Gamma(\propto+1)}-\frac{\ell^{2\propto}}{\Gamma(\propto+1)^{2}}\right\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(n-\mathcal{H}^{-1}\left\{n\mathrm{a}\sqrt{\mathrm{a}}^{\propto+1}-2n\mathrm{a}\sqrt{\mathrm{a}}^{2\propto+1}-\frac{\Gamma(2\propto+1)\mathrm{a}(\sqrt{\mathrm{a}})^{3\propto+1}}{\Gamma(\propto+1)^{2}}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(n-n\frac{\ell^{\propto}}{\Gamma(\propto+1)}+2n\frac{\ell^{\propto+1}}{\Gamma(\propto+2)}+\frac{\ell^{2\propto+2}}{(2\propto+2)\Gamma(\propto+1)^{2}}\right)_{\ell=0},$$

$$=\frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,(2n)_{\ell=0}=\frac{2n\ell^{2\propto}}{\Gamma(2\propto+1)},$$

$$\wp_{3}=\frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{(\wp_{0}+\wp_{1}+\wp_{2})_{nn}-(\wp_{0}+\wp_{1}+\wp_{2})(\wp_{0}+\wp_{1}+\wp_{2})_{n}\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{0-n\left(1-\frac{2\ell^{\propto}}{\Gamma(\propto+1)}+\frac{1\ell^{2\propto}}{\Gamma(\propto+1)^{2}}+\frac{4\ell^{2\propto}}{\Gamma(2\propto+1)}\right)\right\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\left\{\frac{2n\ell^{\propto}}{\Gamma(\propto+1)}-n\left(\frac{1\ell^{2\propto}}{\Gamma(\propto+1)^{2}}+\frac{4\ell^{2\propto}}{\Gamma(2\propto+1)}\right)\right\}\right\}\right)_{\ell=0},$$

$$=\frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(-n\left(\frac{1}{\Gamma(\propto+1)^{2}}+\frac{4}{\Gamma(2\propto+1)}\right)\frac{\Gamma(2\propto+1)}{\Gamma(3\propto+1)}\ell^{3\propto}\right)_{\ell=0},$$

$$= \frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\left(-n\left(\frac{1}{\Gamma(\propto+1)^2}+\frac{4}{\Gamma(2\propto+1)}\right)\Gamma(2\propto+1)\right)_{\ell=0},$$

$$= -n\left(\frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2\Gamma(3\propto+1)}+\frac{4}{\Gamma(3\propto+1)}\right)\ell^{3\propto},$$

$\vdots$

the approximate solution is

$$\wp(n,\ell) = n\left(1-\frac{\ell^{\propto}}{\Gamma(\propto+1)}+\frac{2\ell^{2\propto}}{\Gamma(2\propto+1)}-\left(\frac{\Gamma(2\propto+1)}{\Gamma(\propto+1)^2\Gamma(3\propto+1)}+\frac{4}{\Gamma(3\propto+1)}\right)\ell^{3\propto}+\cdots\right),$$

The exact solution when $\propto = 1$ is

$$\wp_e(n,\ell) = \frac{n}{1+\ell}.$$

Table 8. Approximate solution, exact solution, and absolute errors for (36) with different values of α.

ℓ, n	$\wp_{\propto=0.6}$	$\wp_{\propto=0.7}$	$\wp_{\propto=0.8}$	$\wp_{\propto=0.9}$	$\wp_{\propto=1}$	$\wp_e$	$\lvert\wp_{\propto=1}-\wp_e\rvert$
0.050	0.042489	0.044224	0.045621	0.046743	0.047619	0.047619	0.000000
0.100	0.078255	0.082461	0.085743	0.088523	0.090900	0.090909	0.000009
0.150	0.108335	0.116220	0.121786	0.126356	0.130369	0.130435	0.000066
0.200	0.131982	0.145428	0.154088	0.160701	0.166400	0.166667	0.000267
0.250	0.148029	0.169469	0.182512	0.191697	0.199219	0.200000	0.000781
0.300	0.155132	0.187427	0.206599	0.219229	0.228900	0.230769	0.001869
0.350	0.151869	0.198189	0.225654	0.242975	0.255369	0.259259	0.003891
0.400	0.136768	0.200496	0.238788	0.262416	0.278400	0.285714	0.007314
0.450	0.108339	0.192982	0.244949	0.276864	0.297619	0.310345	0.012726
0.500	0.065078	0.174193	0.242938	0.285463	0.312500	0.333333	0.020833
0.550	0.005479	0.142599	0.231432	0.287207	0.322369	0.354839	0.032470
0.600	-0.071965	0.096613	0.208989	0.280940	0.326400	0.375000	0.048600
0.650	-0.168758	0.034590	0.174061	0.265367	0.323619	0.393939	0.070321
0.700	-0.286397	-0.045158	0.125003	0.239058	0.312900	0.411765	0.098865
0.750	-0.426376	-0.144362	0.060077	0.200451	0.292969	0.428571	0.135603
0.800	-0.590179	-0.264790	-0.022544	0.147856	0.262400	0.444444	0.182044
0.850	-0.779285	-0.408245	-0.124766	0.079462	0.219619	0.459459	0.239841
0.900	-0.995165	-0.576557	-0.248573	-0.006665	0.162900	0.473684	0.310784
0.950	-1.239282	-0.771589	-0.396023	-0.112576	0.090369	0.487179	0.396811
1.000	-1.513091	-0.995225	-0.569247	-0.240438	0.000000	0.500000	0.500000

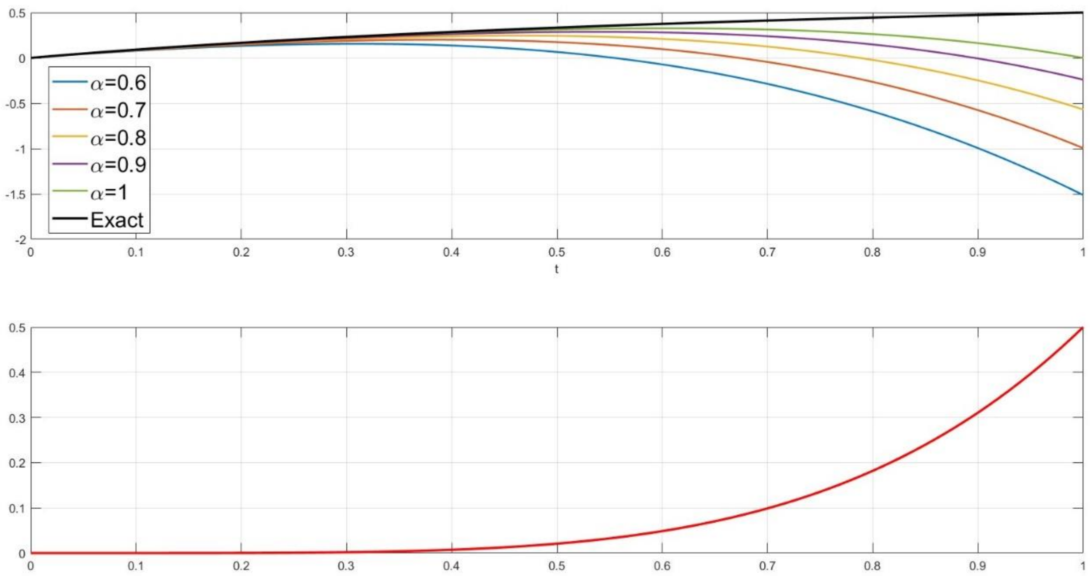

Figure 8. Approximate solution, exact solution, and absolute errors for (36) with different values of α

Example. 4.2.3 Consider the following time–fractional nonlinear dispersive partial differential equation of Caputo type.

$$ {}^{c}D_{\ell}^{\alpha}\wp(n,\ell) - 6\wp(n,\ell)\wp_{n}(n,\ell) + \wp_{nnn}(n,\ell) = 0, \qquad 0 < \alpha \leq 1 \tag{40}$$

with initial conditions, $\wp(n,0) = 6n$.

The Yasser Jassim transform and taking Yasser Jassim inverse on both sides of (40), we obtain

$$\wp(n,\ell) = 6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6\wp(n,\ell)\wp_{n}(n,\ell) - \wp_{nnn}(n,\ell)\}\right\}, \tag{41}$$

Using the Hussein–Jassim framework, Eq. (41) is recast into the corresponding recursive integral representation,

$$\wp(n,\ell) = \sum_{i=0}^{\infty}\frac{\ell^{i\alpha}}{\Gamma(i\alpha+1)}\, D_{\ell}^{i\alpha}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6\wp(n,\ell)\wp_{n}(n,\ell) - \wp_{nnn}(n,\ell)\}\right\}\right)_{\ell=0}. \tag{42}$$

Assume that.

$$\wp(n,\ell) = \sum_{i=0}^{\infty}\wp_{i},$$

By substituting in (42).

$$\sum_{i=0}^{\infty}\wp_{i} = \sum_{i=0}^{\infty}\frac{\ell^{i\alpha}}{\Gamma(i\alpha+1)}\, D_{\ell}^{i\alpha}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{6\left(\left(\sum_{j=0}^{i-1}\wp_{i}\right)\left(\sum_{j=0}^{i-1}\wp_{i}\right)_{n}\right) - \left(\sum_{j=0}^{i-1}\wp_{i}\right)_{nnn}\right\}\right\}\right)_{\ell=0}. \tag{43}$$

By comparing

$$\wp_{0} = \wp(n,0) = 6n$$

$$\wp_{1} = \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{6\left((\wp_{0})(\wp_{0})_{n}\right) - (\wp_{0})_{nnn}\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{6\left((6n)(6n)_{n}\right) - (6n)_{nnn}\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\left\{6\left((6n)(6)\right) - (0)\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{a}^{\alpha}\mathcal{H}\{6^{3}(n)\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}\left(6n + \mathcal{H}^{-1}\left\{6^{3}n a\sqrt{a}^{\alpha+1}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}\left(6n + 6^{3}n\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right)_{\ell=0},$$

$$= \frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\, {}^{c}D_{\ell}^{\alpha}(0 + 6^{3}n)_{\ell=0},$$

$$= n\frac{6^3\ell^{\propto}}{\Gamma(\propto+1)},$$

$$\wp_2 = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{6\big((\wp_0+\wp_1)(\wp_0+\wp_1)_n\big) - (\wp_0+\wp_1)_{nnn}\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{6\left(\left(6n + n\frac{6^3\ell^{\propto}}{\Gamma(\propto+1)}\right)\left(6+\frac{6^3\ell^{\propto}}{\Gamma(\propto+1)}\right)\right) - (0)_{nnn}\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{6^3 n + \frac{2n6^5\ell^{\propto}}{\Gamma(\propto+1)} + \frac{6^7\ell^{2\propto}}{\Gamma(\propto+1)^2}\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(6n + \mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{6^3 n\mathfrak{a}\sqrt{\mathfrak{a}} + 2n6^5\mathfrak{a}\sqrt{\mathfrak{a}}^{\propto+1} + \frac{6^7\Gamma(2\propto+1)\mathfrak{a}(\sqrt{\mathfrak{a}})^{2\propto+1}}{\Gamma(\propto+1)^2}\right\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(6n + \mathcal{H}^{-1}\left\{6^3 n\mathfrak{a}\sqrt{\mathfrak{a}}^{\propto} + 2n6^5\mathfrak{a}\sqrt{\mathfrak{a}}^{2\propto+1} + \frac{6^7\Gamma(2\propto+1)\mathfrak{a}(\sqrt{\mathfrak{a}})^{3\propto+1}}{\Gamma(\propto+1)^2}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,{}^{c}D_{\ell}^{2\propto}\left(6n + 6^3 n\frac{\ell^{\propto}}{\Gamma(\propto+1)} + 2n6^5\frac{\ell^{2\propto}}{\Gamma(2\propto+1)} + \frac{6^7\ell^{3\propto}}{\Gamma(\propto+1)^2}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}\,(2n6^5)_{\ell=0},$$

$$= \frac{2n6^5\ell^{2\propto}}{\Gamma(2\propto+1)},$$

$$\wp_3 = \frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{\begin{matrix}6\big((\wp_0+\wp_1+\wp_2)(\wp_0+\wp_1+\wp_2)_n\big)\\ -(\wp_0+\wp_1+\wp_2)_{nnn}\end{matrix}\right\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathfrak{a}}^{\propto}\mathcal{H}\left\{6n\left(6+\frac{(6^3)^2}{\Gamma(\propto+1)^2}+\frac{24(6^5)}{\Gamma(2\propto+1)}\right)\ell^{2\propto}\right\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(\mathcal{H}^{-1}\left\{6n\left(\frac{(6^3)^2}{\Gamma(\propto+1)^2}+\frac{24(6^5)}{\Gamma(2\propto+1)}\right)\mathfrak{a}\sqrt{\mathfrak{a}}^{2\propto+1}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\,{}^{c}D_{\ell}^{3\propto}\left(6n\left(\frac{(6^3)^2}{\Gamma(\propto+1)^2}+\frac{24(6^5)}{\Gamma(2\propto+1)}\right)\frac{\Gamma(2\propto+1)}{\Gamma(3\propto+1)}\ell^{3\propto}\right)_{\ell=0}$$

$$= \frac{\ell^{3\propto}}{\Gamma(3\propto+1)}\left(6n\left(\frac{(6^3)^2}{\Gamma(\propto+1)^2}+\frac{24(6^5)}{\Gamma(2\propto+1)}\right)\frac{\Gamma(2\propto+1)}{\Gamma(3\propto+1)}\Gamma(3\propto+1)\right)_{\ell=0}$$

$$= 6n\left(\frac{(6^3)^2\Gamma(2\propto+1)}{\Gamma(\propto+1)^2\Gamma(3\propto+1)}+\frac{24(6^5)}{\Gamma(3\propto+1)}\right)\ell^{3\propto}$$

$$= \left(\frac{6^7\Gamma(2\propto+1)}{\Gamma(\propto+1)^2\Gamma(3\propto+1)}+\frac{24(6^6)}{\Gamma(3\propto+1)}\right)n\ell^{3\propto}$$

$$\vdots$$

the approximate solution is

$$\wp(n,\ell) = \sum_{i=0}^{\infty}\wp_i = \wp_0+\wp_1+\wp_2+\wp_3+\cdots$$

$$\wp(n,\ell) = 6n + n\frac{6^3\ell^{\propto}}{\Gamma(\propto+1)} + 2n\frac{6^5\ell^{2\propto}}{\Gamma(2\propto+1)} + \left(\frac{6^7\Gamma(2\propto+1)}{\Gamma(\propto+1)^2\Gamma(3\propto+1)}+\frac{24(6^6)}{\Gamma(3\propto+1)}\right)n\ell^{3\propto}+\cdots$$

The exact solution when $\propto = 1$ is

$$\wp(n,\ell) = \frac{6n}{1-6^2\ell}.$$

Table 9. Approximate solution, exact solution, and absolute errors for (40) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1}-\wp_e\rvert$
0.001	0.013973	0.008737	0.007101	0.006486	0.006224	0.006224	0.000000
0.002	0.044057	0.022807	0.016319	0.013936	0.012927	0.012931	0.000004
0.003	0.092890	0.043021	0.027904	0.022444	0.020158	0.020179	0.000022
0.004	0.162973	0.070245	0.042154	0.032120	0.027966	0.028037	0.000072
0.005	0.256615	0.105323	0.059377	0.043082	0.036401	0.036585	0.000184
0.006	0.375981	0.149078	0.079881	0.055451	0.045516	0.045918	0.000402
0.007	0.523126	0.202317	0.103974	0.069350	0.055363	0.056150	0.000787
0.008	0.700018	0.265831	0.131968	0.084903	0.065996	0.067416	0.001419
0.009	0.908549	0.340402	0.164173	0.102239	0.077471	0.079882	0.002411
0.010	1.150551	0.426802	0.200901	0.121485	0.089843	0.093750	0.003907
0.011	1.427804	0.525794	0.242467	0.142773	0.103169	0.109272	0.006103
0.012	1.742040	0.638138	0.289186	0.166235	0.117508	0.126761	0.009252
0.013	2.094952	0.764585	0.341374	0.192007	0.132920	0.146617	0.013696
0.014	2.488195	0.905884	0.399352	0.220225	0.149466	0.169355	0.019889
0.015	2.923394	1.062779	0.463439	0.251026	0.167206	0.195652	0.028446
0.016	3.402141	1.236009	0.533958	0.284552	0.186204	0.226415	0.040211
0.017	3.926004	1.426314	0.611233	0.320944	0.206524	0.262887	0.056362
0.018	4.496524	1.634428	0.695592	0.360347	0.228231	0.306818	0.078587
0.019	5.115220	1.861084	0.787363	0.402905	0.251392	0.360759	0.109368
0.020	5.783590	2.107012	0.886876	0.448767	0.276073	0.428571	0.152498

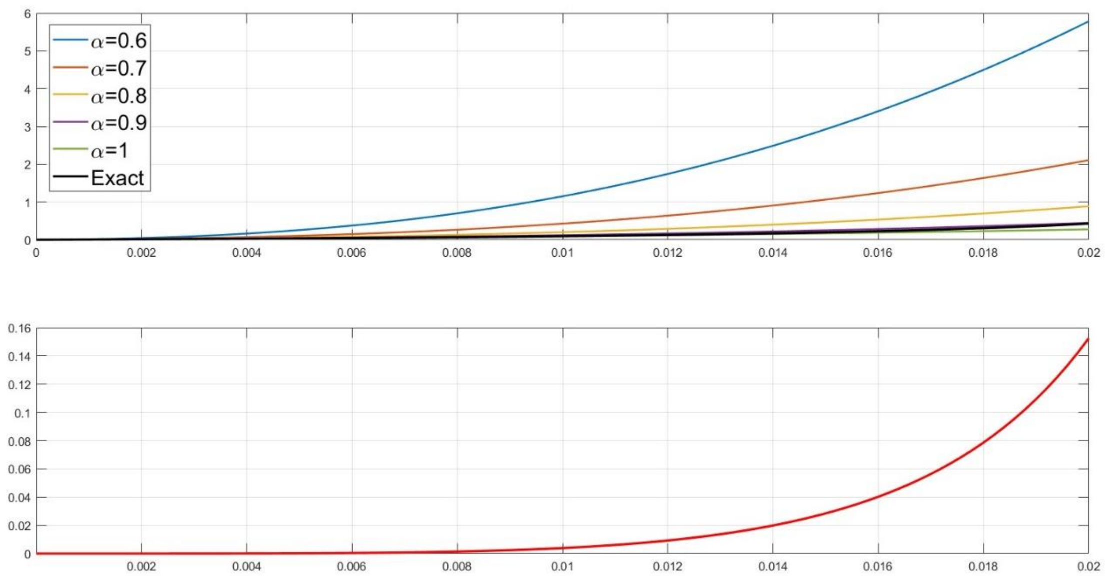

Figure 9. Approximate solution, exact solution, and absolute errors for (40) with different values of α

Example. 4.2.4 Consider the following system of coupled time–fractional partial differential equations of Caputo type.

$$\begin{aligned} {}^{c}D_{\ell}^{\alpha}\wp(\varrho,n,\ell) - \upsilon_{\varrho}(\varrho,n,\ell)\omega_{n}(\varrho,n,\ell) &= 1, \qquad 0<\alpha\leq 1\\ {}^{c}D_{\ell}^{\alpha}\upsilon(\varrho,n,\ell) - \omega_{\varrho}(\varrho,n,\ell)\wp_{n}(\varrho,n,\ell) &= 5, \qquad 0<\alpha\leq 1\\ {}^{c}D_{\ell}^{\alpha}\omega(\varrho,n,\ell) - \wp_{\varrho}(\varrho,n,\ell)\upsilon_{n}(\varrho,n,\ell) &= 5, \qquad 0<\alpha\leq 1 \end{aligned} \tag{44}$$

with initial conditions, $\wp(\varrho,n,0)=\varrho+2n$, $\upsilon(\varrho,n,0)=\varrho-2n$, $\omega(\varrho,n,0)=-\varrho+2n$.
The Yasser Jassim transform and taking Yasser Jassim inverse on both sides of (44), we obtain

$$\wp(\varrho,n,\ell)=\mathcal{H}^{-1}\left\{\sqrt{\text{a}}\,\mathcal{H}\{\varrho+2n\}\right\}+\mathcal{H}^{-1}\left\{\sqrt{\text{a}}^{\alpha}\mathcal{H}\{1+\upsilon_{\varrho}(\varrho,n,\ell)\omega_{n}(\varrho,n,\ell)\}\right\},$$

$$\upsilon(\varrho,n,\ell) = \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho-2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+\omega_{\varrho}(\varrho,n,\ell)\wp_{n}(\varrho,n,\ell)\}\right\}, \quad (45)$$
$$\omega(\varrho,n,\ell) = \mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{-\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+\wp_{\varrho}(\varrho,n,\ell)\upsilon_{n}(\varrho,n,\ell)\}\right\},$$

Using the Hussein–Jassim framework, Eq. (45) is recast into the corresponding recursive integral representation,

$$\wp(\varrho,n,\ell) = \sum_{i=0}^{\infty}\frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{1+\upsilon_{\varrho}(\varrho,n,\ell)\omega_{n}(\varrho,n,\ell)\}\right\}\right)_{\ell=0},$$
$$\upsilon(\varrho,n,\ell) = \sum_{i=0}^{\infty}\frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho-2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+\omega_{\varrho}(\varrho,n,\ell)\wp_{n}(\varrho,n,\ell)\}\right\}\right)_{\ell=0}, \quad (46)$$
$$\omega(\varrho,n,\ell) = \sum_{i=0}^{\infty}\frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{-\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+\wp_{\varrho}(\varrho,n,\ell)\upsilon_{n}(\varrho,n,\ell)\}\right\}\right)_{\ell=0},$$

Assume that

$$\wp(\varrho,n,\ell) = \sum_{i=0}^{\infty}\wp_i,$$
$$\upsilon(\varrho,n,\ell) = \sum_{i=0}^{\infty}\upsilon_i$$
$$\omega(\varrho,n,\ell) = \sum_{i=0}^{\infty}\omega_i$$

by substituting in (46).

$$\sum_{i=0}^{\infty}\wp_i = \sum_{i=0}^{\infty}\frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{1+(\sum_{i=0}^{\infty}\upsilon_{i\varrho})(\sum_{i=0}^{\infty}\omega_{in})\}\right\}\right)_{\ell=0},$$
$$\sum_{i=0}^{\infty}\upsilon_i = \sum_{i=0}^{\infty}\frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho-2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+(\sum_{i=0}^{\infty}\omega_{i\varrho})(\sum_{i=0}^{\infty}\wp_{in})\}\right\}\right)_{\ell=0}, \quad (47)$$
$$\sum_{i=0}^{\infty}\omega_i = \sum_{i=0}^{\infty}\frac{\ell^{i\propto}}{\Gamma(i\propto+1)} D_{\ell}^{i\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{-\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+(\sum_{i=0}^{\infty}\wp_{i\varrho})(\sum_{i=0}^{\infty}\upsilon_{in})\}\right\}\right)_{\ell=0},$$

By comparing

$$\wp_0 = \varrho + 2n$$
$$\upsilon_0 = \varrho - 2n$$
$$\omega_0 = -\varrho + 2n$$

$$\wp_1 = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{1+(\upsilon_{0\varrho})(\omega_{0n})\}\right\}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho+2n+\mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{1+(1)(2)\}\right\}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho+2n+\mathcal{H}^{-1}\left\{3a\sqrt{a}^{\propto+1}\right\}\right)_{\ell=0} = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho+2n+3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho+2n+3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0} = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}(3)_{\ell=0},$$
$$= 3\frac{\ell^{\propto}}{\Gamma(\propto+1)},$$
$$\upsilon_1 = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{\varrho-2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+(\omega_{0\varrho})(\wp_{0n})\}\right\}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho-2n+\mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5-2\}\right\}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho-2n+\mathcal{H}^{-1}\left\{3a\sqrt{a}^{\propto+1}\right\}\right)_{\ell=0} = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho-2n+3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\varrho-2n+3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)}(3)_{\ell=0} = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)},$$
$$\omega_1 = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{a}\,\mathcal{H}\{-\varrho+2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5+(\wp_{0\varrho})(\upsilon_{0n})\}\right\}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(-\varrho+2n+\mathcal{H}^{-1}\left\{\sqrt{a}^{\propto}\mathcal{H}\{5-2\}\right\}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(-\varrho+2n+\mathcal{H}^{-1}\left\{3a\sqrt{a}^{\propto+1}\right\}\right)_{\ell=0} = \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(-\varrho+2n+3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0},$$
$$= \frac{\ell^{\propto}}{\Gamma(\propto+1)} D_{\ell}^{\propto}\left(-\varrho+2n+3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0} = \frac{\ell^{\propto}}{\Gamma(\propto+1)}(3)_{\ell=0},$$

$$= 3\frac{\ell^{\propto}}{\Gamma(\propto+1)},$$

$$\wp_2 = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}\,\mathcal{H}\{\varrho + 2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{1 + (\upsilon_0+\upsilon_1)_{\varrho}(\omega_0+\omega_1)_{n}\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\varrho + 2n + \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{1+2\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\varrho + 2n + 3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0} = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}(0)_{\ell=0},$$

$$= 0$$

$$\upsilon_2 = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}\,\mathcal{H}\{\varrho - 2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{5 + (\omega_0+\omega_1)_{\varrho}(\wp_0+\wp_1)_{n}\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\varrho - 2n + \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{5-2\}\right\}\right)_{\ell=0},$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\varrho - 2n + 3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0} = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}(0)_{\ell=0},$$

$$= 0,$$

$$\omega_2 = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}\,\mathcal{H}\{\varrho - 2n\}\right\} + \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{2 + (\wp_0+\wp_1)_{\varrho}(\upsilon_0+\upsilon_1)_{n}\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\varrho - 2n + \mathcal{H}^{-1}\left\{\sqrt{\mathrm{a}}^{\propto}\mathcal{H}\{5-2\}\right\}\right)_{\ell=0}$$

$$= \frac{\ell^{2\propto}}{\Gamma(2\propto+1)} {}^{c}D_{\ell}^{2\propto}\left(\varrho - 2n + 3\frac{\ell^{\propto}}{\Gamma(\propto+1)}\right)_{\ell=0} = \frac{\ell^{2\propto}}{\Gamma(2\propto+1)}(0)_{\ell=0},$$

$$= 0$$

$$\vdots$$

Consequently, the approximate solution is

$$\wp(\varrho, n, \ell) = \sum_{\mathrm{i}=0}^{\infty} \wp_{\mathrm{i}} = \wp_0 + \wp_1 + \wp_2 + \cdots$$

$$\wp_{\mathrm{a}}(\varrho, n, \ell) = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)} + \varrho + 2n.$$

$$\upsilon(\varrho, n, \ell) = \sum_{\mathrm{i}=0}^{\infty} \upsilon_{\mathrm{i}} = \upsilon_0 + \upsilon_1 + \upsilon_2 + \cdots$$

$$\upsilon_{\mathrm{a}}(\varrho, n, \ell) = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)} + \varrho - 2n.$$

$$\omega(\varrho, n, \ell) = \sum_{\mathrm{i}=0}^{\infty} \omega_{\mathrm{i}} = \omega_0 + \omega_1 + \omega_2 + \cdots$$

$$\omega_{\mathrm{a}}(\varrho, n, \ell) = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)} - \varrho + 2n.$$

The exact solution

$$\wp_e(\varrho, n, \ell) = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)} + \varrho + 2n.$$

$$\upsilon_e(\varrho, n, \ell) = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)} + \varrho - 2n.$$

$$\omega_e(\varrho, n, \ell) = 3\frac{\ell^{\propto}}{\Gamma(\propto+1)} - \varrho + 2n.$$

Table 10. Approximate solution, exact solution, and absolute errors for (44) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1}-\wp_e\rvert$
0.050	1.656418	1.505517	1.393203	1.310438	1.250000	1.250000	0.000000
0.100	2.043372	1.858764	1.710496	1.592692	1.500000	1.500000	0.000000
0.150	2.375657	2.174971	2.006099	1.865632	1.750000	1.750000	0.000000
0.200	2.678313	2.470166	2.288826	2.132789	2.000000	2.000000	0.000000
0.250	2.961448	2.751088	2.562538	2.395773	2.250000	2.250000	0.000000
0.300	3.230392	3.021395	2.829390	2.655507	2.500000	2.500000	0.000000
0.350	3.488381	3.283353	3.090743	2.912588	2.750000	2.750000	0.000000
0.400	3.737560	3.538489	3.347535	3.167433	3.000000	3.000000	0.000000
0.450	3.979442	3.787899	3.600445	3.420349	3.250000	3.250000	0.000000
0.500	4.215140	4.032399	3.849987	3.671571	3.500000	3.500000	0.000000
0.550	4.445508	4.272621	4.096562	3.921287	3.750000	3.750000	0.000000
0.600	4.671212	4.509064	4.340492	4.169645	4.000000	4.000000	0.000000
0.650	4.892789	4.742133	4.582040	4.416771	4.250000	4.250000	0.000000
0.700	5.110679	4.972164	4.821425	4.662769	4.500000	4.500000	0.000000
0.750	5.325244	5.199435	5.058831	4.907726	4.750000	4.750000	0.000000
0.800	5.536792	5.424184	5.294416	5.151719	5.000000	5.000000	0.000000
0.850	5.745584	5.646614	5.528315	5.394815	5.250000	5.250000	0.000000
0.900	5.951844	5.866901	5.760647	5.637071	5.500000	5.500000	0.000000
0.950	6.155768	6.085199	5.991516	5.878538	5.750000	5.750000	0.000000
1.000	6.357525	6.301642	6.221014	6.119262	6.000000	6.000000	0.000000

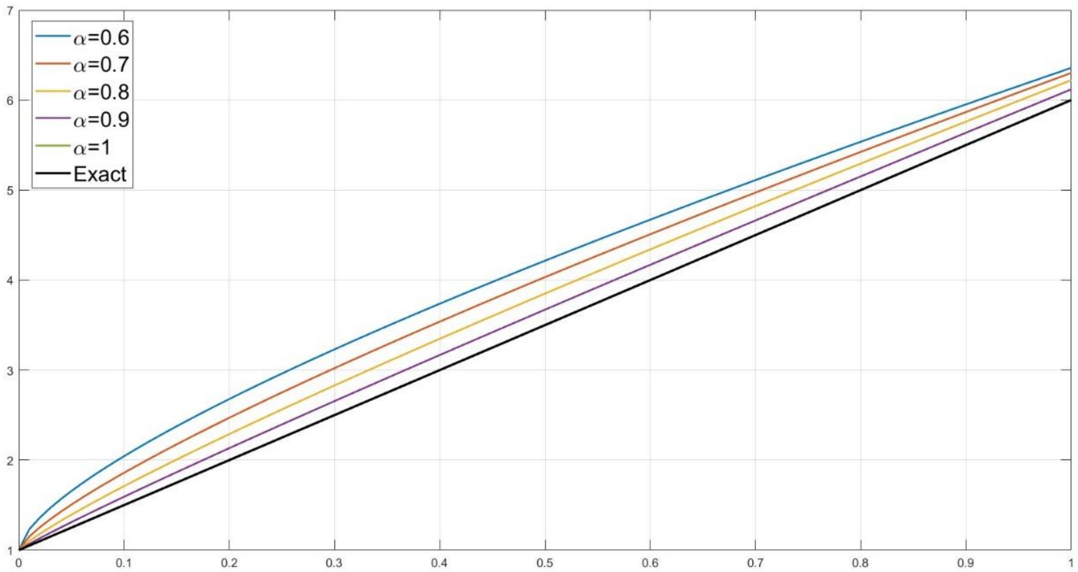

Figure 10. Approximate solution, exact solution, and absolute errors for (44) with different values of α

Table 11. Approximate solution, exact solution, and absolute errors for (44) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1}-\wp_e\rvert$
0.050	1.456418	1.305517	1.193203	1.110438	1.050000	1.050000	0.000000
0.100	1.643372	1.458764	1.310496	1.192692	1.100000	1.100000	0.000000
0.150	1.775657	1.574971	1.406099	1.265632	1.150000	1.150000	0.000000
0.200	1.878313	1.670166	1.488826	1.332789	1.200000	1.200000	0.000000
0.250	1.961448	1.751088	1.562538	1.395773	1.250000	1.250000	0.000000
0.300	2.030392	1.821395	1.629390	1.455507	1.300000	1.300000	0.000000
0.350	2.088381	1.883353	1.690743	1.512588	1.350000	1.350000	0.000000
0.400	2.137560	1.938489	1.747535	1.567433	1.400000	1.400000	0.000000
0.450	2.179442	1.987899	1.800445	1.620349	1.450000	1.450000	0.000000
0.500	2.215140	2.032399	1.849987	1.671571	1.500000	1.500000	0.000000
0.550	2.245508	2.072621	1.896562	1.721287	1.550000	1.550000	0.000000
0.600	2.271212	2.109064	1.940492	1.769645	1.600000	1.600000	0.000000
0.650	2.292789	2.142133	1.982040	1.816771	1.650000	1.650000	0.000000
0.700	2.310679	2.172164	2.021425	1.862769	1.700000	1.700000	0.000000
0.750	2.325244	2.199435	2.058831	1.907726	1.750000	1.750000	0.000000
0.800	2.336792	2.224184	2.094416	1.951719	1.800000	1.800000	0.000000
0.850	2.345584	2.246614	2.128315	1.994815	1.850000	1.850000	0.000000
0.900	2.351844	2.266901	2.160647	2.037071	1.900000	1.900000	0.000000
0.950	2.355768	2.285199	2.191516	2.078538	1.950000	1.950000	0.000000
1.000	2.357525	2.301642	2.221014	2.119262	2.000000	2.000000	0.000000

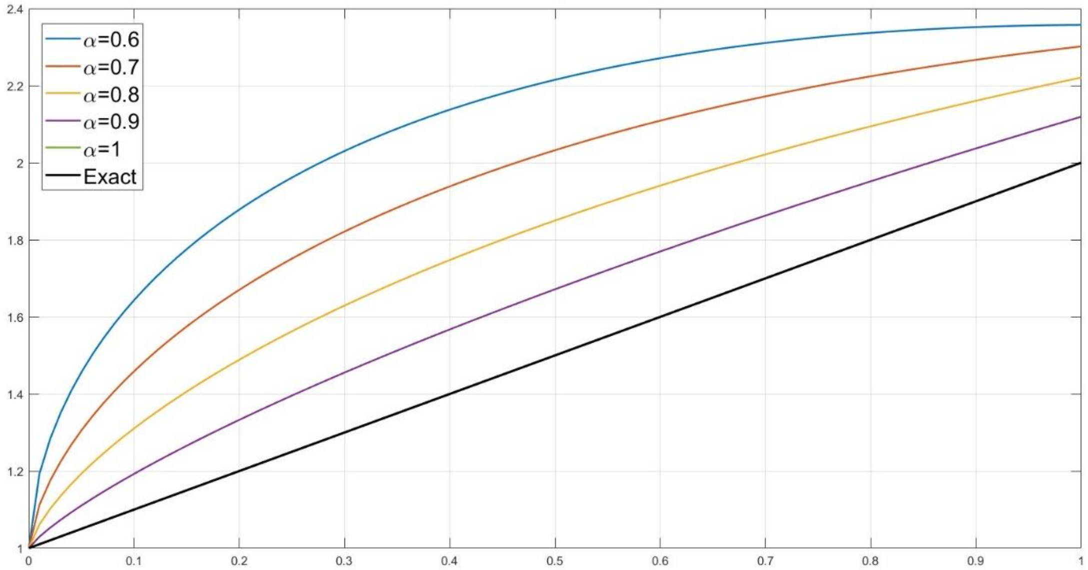

Figure 11. Approximate solution, exact solution, and absolute errors for (44) with different values of α

Table 12. Approximate solution, exact solution, and absolute errors for (44) with different values of α.

ℓ, n	$\wp_{\alpha=0.6}$	$\wp_{\alpha=0.7}$	$\wp_{\alpha=0.8}$	$\wp_{\alpha=0.9}$	$\wp_{\alpha=1}$	$\wp_e$	$\lvert\wp_{\alpha=1}-\wp_e\rvert$
0.050	-0.343582	-0.494483	-0.606797	-0.689562	-0.750000	-0.750000	0.000000
0.100	0.043372	-0.141236	-0.289504	-0.407308	-0.500000	-0.500000	0.000000
0.150	0.375657	0.174971	0.006099	-0.134368	-0.250000	-0.250000	0.000000
0.200	0.678313	0.470166	0.288826	0.132789	0.000000	0.000000	0.000000
0.250	0.961448	0.751088	0.562538	0.395773	0.250000	0.250000	0.000000
0.300	1.230392	1.021395	0.829390	0.655507	0.500000	0.500000	0.000000
0.350	1.488381	1.283353	1.090743	0.912588	0.750000	0.750000	0.000000
0.400	1.737560	1.538489	1.347535	1.167433	1.000000	1.000000	0.000000
0.450	1.979442	1.787899	1.600445	1.420349	1.250000	1.250000	0.000000
0.500	2.215140	2.032399	1.849987	1.671571	1.500000	1.500000	0.000000
0.550	2.445508	2.272621	2.096562	1.921287	1.750000	1.750000	0.000000
0.600	2.671212	2.509064	2.340492	2.169645	2.000000	2.000000	0.000000
0.650	2.892789	2.742133	2.582040	2.416771	2.250000	2.250000	0.000000
0.700	3.110679	2.972164	2.821425	2.662769	2.500000	2.500000	0.000000
0.750	3.325244	3.199435	3.058831	2.907726	2.750000	2.750000	0.000000
0.800	3.536792	3.424184	3.294416	3.151719	3.000000	3.000000	0.000000
0.850	3.745584	3.646614	3.528315	3.394815	3.250000	3.250000	0.000000
0.900	3.951844	3.866901	3.760647	3.637071	3.500000	3.500000	0.000000
0.950	4.155768	4.085199	3.991516	3.878538	3.750000	3.750000	0.000000
1.000	4.357525	4.301642	4.221014	4.119262	4.000000	4.000000	0.000000

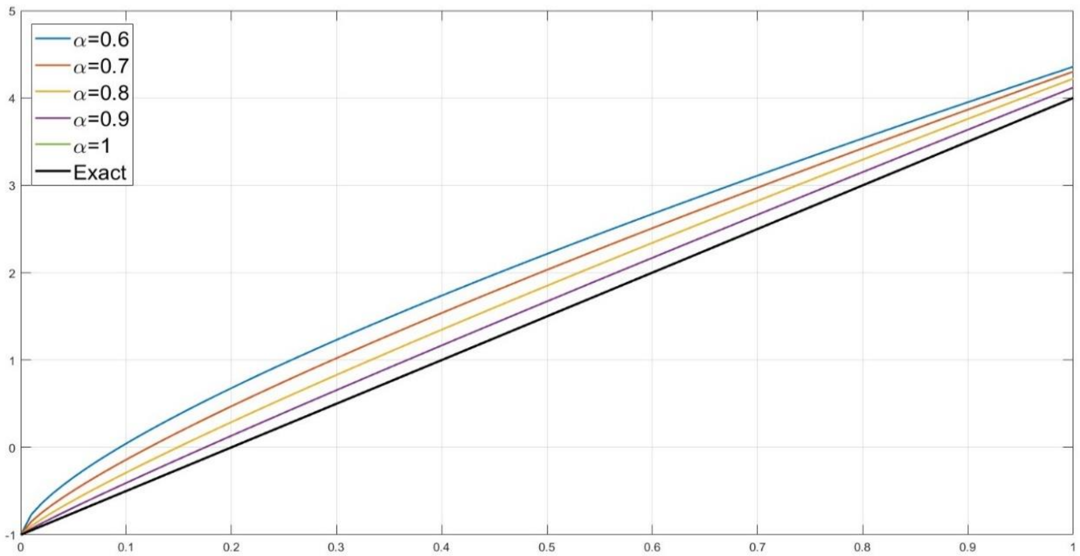

Figure 12. Approximate solution, exact solution, and absolute errors for (44) with different values of α

5- Conclusion Section

Two methods have been created to analyze time-fractional PDEs in the Caputo sense by combining iterative series methods with the Yasser–Jassim transform. In the first scheme, the transform provides a convenient route to the homotopy perturbation expansion; in the second, it supports a Hussein–Jassim recursive evaluation of the solution components. The computed results indicate that inserting the transform does not affect the convergence trend of the original iterative method, and the obtained approximations remain consistent with the corresponding exact solutions in the integer-order limit. The main benefit of the transform is a reduction in algebraic workload and a clearer iterative structure, which is consistent with other successful transform-assisted homotopy approaches reported in the literature [11,15]. The Yasser-Jassim transforms are also emphasized for their simplifying role in solving differential and integral models [20].

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