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ON CONVERGENCE OF THE TRIPLE LAPLACE HOMOTOPY PERTURBATION METHOD OF SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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Abstract:

In this paper, we combine two methods, triple Laplace transforms and Homotopy Perturbation Method (TLHPM)to solve the problems and mixing problems of science and engineering. (TLHPM) is used to realize accurate and perfect solutions of nonlinear partial differential equations (NLPDEs) subject to the initial conditions. Also, in this work, we are particularly interested in nonlinear partial differential equations with their initial conditions. Finally, the triple Laplace Homotopy Perturbation applied to find the solutions of nonlinear partial differential equations in three dimensions; also, we study some Convergence theorems and some properties.

Keywords: Triple Laplace Transform, Convergence Triple Laplace Transform, Homotopy Perturbation Method, nonlinear partial differential equations.

1-IntroductionSection

In recent years, many authors have devoted their attention to studying solutions of nonlinear partial differential equations using various methods. Among these attempts are the Adomian decomposition method, homotopy perturbation method, variational iteration method [1–5], Laplace variational iteration method [6–9], differential transform method, reduced differential transform method, and projected differential transform method.

P. S. Laplace (1749–1827) introduced the idea of the Laplace transform in 1782 [15]. The Laplace transform, denoted by the operator, L is defined as:

$$L\left[f(\tau):\rho)\right]=\int_0^\infty e^{-\rho\tau}f(\tau)d\tau \ , \quad \tau>0 \tag{1}$$

Where $\tau>0$ and $\rho=a+ib, i=\sqrt{-1}$.

The main objective of the study is to extend the method of Homotopy Perturbation algorithm to convert the one-dimensional triple Laplace Homotopy Perturbation method to three-dimensional triple Laplace Homotopy Perturbation methods to solve linear and nonlinear partial differential equations. The advantage of this method is Its capability to combine two powerful methods to obtain accurate solutions and study the convergence of obtained solutions to linear and nonlinear three-dimensional equations. Several examples are given to reestablish the effectiveness of this method.[14].

2-Definitions and Theorems of Triple Laplace Transform [10-13]

Definition 2.1: Let $f(\chi,\gamma,\tau)$ be a function that can be expressed as a convergent infinite series, and let $(\chi,\gamma,\tau)\in R_3^+$, then, the triple Laplace transform is denoted by:

$$L_3\left[f(\chi,\gamma,\tau):(\sigma,\rho,\nu)\right]=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\sigma\chi-\rho\gamma-\nu\tau}f(\chi,\gamma,\tau)d\chi d\gamma d\tau, \qquad (2)$$

Where, $\chi,\gamma,\tau>0$ and σ,ρ,ν are Laplace variables, and

$$f(\chi,\gamma,\tau)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\sigma\chi}[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}e^{\rho\gamma}[\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{\nu\tau}f(\sigma,\rho,\nu)d\nu]d\rho]d\sigma,$$

is the inverse triple Laplace transform denoted by: L_3^{-1}.

Theorem 2.2: let $F_n(\rho)$ denote Laplace transform of n^th^ derivative, $f^{\,n}(\tau)$ of $f(\tau)$, then: for $n\geq 1$,

$$F_n(\rho)=\rho^n F(\rho)-\sum_{k=0}^{n-1}\rho^{n-1-k}f^{(k)}(0), \qquad (3)$$

To obtain triple Laplace transform of partial derivative we use integration by parts, and then we have:

$$L_3\left[\frac{\partial f(\chi,\gamma,\tau)}{\partial\tau}\right]=\nu F(\sigma,\rho,\nu)-f(\chi,\gamma,0),$$

$$L_3\left[\frac{\partial^2 f(\chi,\gamma,\tau)}{\partial\tau^2}\right]=\nu^2F(\sigma,\rho,\nu)-\nu F(\sigma,\rho,0)-\frac{\partial f(\chi,\gamma,0)}{\partial\tau}, \qquad (4)$$

Convolution Property: [16]

If single Laplace transform and double Laplace transform convolution theorems becomes of $F(\chi,\gamma), G(\tau)\,\&\,F(\chi,\gamma,\tau)$ with respect to t are given by

$L_{\gamma\tau}\{F(\chi,\tau)\}=f(\rho,\nu), L_{\gamma\tau}\{F(\chi,\gamma,\tau)\}=f(\chi,\rho,\nu)\,\&\,L_\tau\{G(\tau)\}=g(\nu)$ then,

$$L_\tau\{G(\tau)*F(\chi,\tau)\}=L_\tau\{G(\tau)\}L_\tau\{F(\chi,\tau)\}=g(\nu)f(\chi,\nu)$$

Where $G(\tau)*F(\chi,\tau)=\int_0^\tau G(\tau-t)F(\chi,\tau)dt$. and

$$L_{\gamma\tau}\{G(\tau)*F(\chi,\gamma,\tau)\}=L_{\gamma\tau}\{G(\tau)\}L_{\gamma\tau}\{F(\chi,\gamma,\tau)\}=g(\nu)f(\chi,\rho,\nu)$$

Therefore, by using double Laplace transform convolution theorem becomes

$$\begin{aligned}L_{\gamma\tau}\{G(\tau)*F(\chi,\gamma,\tau)\}&=L_{\gamma\tau}\left\{\int_0^\tau G(\tau-t)F(\chi,\gamma,\tau)dt\right\}\\&=g(\nu)f(\sigma,\gamma,\nu).\end{aligned}$$

$$L_{\chi\gamma\tau}\left\{\int_0^\tau G(\tau-t)F(\chi,\gamma,\tau)dt\right\}=L_{\chi\gamma\tau}\{G(\tau)*F(\chi,\gamma,\tau)\}=g(\nu)f(\sigma,\rho,\nu)$$

3- Triple Laplace Homotopy Perturbation Method for Linear PDEs[17-18]

To illustrate the basic idea of this method, we consider a general linear nonhomogeneous partial differential equation,

$$Ru(\chi,\gamma,\tau)+Qu(\chi,\gamma,\tau)=g(\chi,\gamma,\tau), \tag{5}$$

with the initial condition,

$$u(\chi,\gamma,0)=f(\chi,\gamma), \tag{6}$$

where $Ru(\chi,\gamma,\tau)=\beta\dfrac{\partial u(\chi,\gamma,\tau)}{\partial\tau}$ is the partial derivative of the function $u(\chi,\gamma,\tau)$ of the first order, Q represents the general linear differential operator, and $g(\chi,\gamma,\tau)$ is the source term.

Applying the triple Laplace transform (denoted in this paper by L_3) on both sides of Eq. (5), we get:

$$L_3\left[Ru(\chi,\gamma,\tau)\right]+L_3\left[Qu(\chi,\gamma,\tau)\right]=L_3\left[g(\chi,\gamma,\tau)\right], \tag{7}$$

By using the properties of triple Laplace transform, we obtain:

$$\beta v L_3\left[u(\chi,\gamma,\tau)\right]=\beta f(\chi,\gamma)+L_3\left[g(\chi,\gamma,\tau)\right]-L_3\left[Qu(\chi,\gamma,\tau)\right], \tag{8}$$

we have

$$L_3\left[u(\chi,\gamma,\tau)\right]=v^{-1}f(\chi,\gamma)+\beta^{-1}v^{-1}L_3\left[g(\chi,\gamma,\tau)\right]-\beta^{-1}v^{-1}L_3\left[Qu(\chi,\gamma,\tau)\right], \tag{9}$$

Operating the inverse triple Laplace transform on both sides of Eq. (9), to get:

$$u(\chi,\gamma,\tau)=L_3^{-1}\left\{v^{-1}f(\chi,\gamma)\right\}+L_3^{-1}\left\{\beta^{-1}v^{-1}L_3\left[g(\chi,\gamma,\tau)\right]\right\}-L_3^{-1}\left\{\beta^{-1}v^{-1}L_3\left[Qu(\chi,\gamma,\tau)\right]\right\}, \tag{10}$$

Now, we apply the Homotopy perturbation method

$$u(\chi,\gamma,\tau)=\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau), \tag{11}$$

Substituting equation (11) into (10), to obtain:

$$\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau)=G(\chi,\gamma,\tau)-p\left\{L_3^{-1}\left(\beta^{-1}v^{-1}L_3\left[\sum_{n=0}^{\infty}p^{n}Q_{n}(\chi,\gamma,\tau)\right]\right)\right\}, \tag{12}$$

Where $G(\chi,\gamma,\tau)$ represents the term arising from the source term and the prescribed initial conditions.

This is the coupling of the triple Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficients of the like power of p, the following approximations are obtained:

$$\begin{aligned}
&p^{0}: u_{0}(\chi,\gamma,\tau)=G(\chi,\gamma,\tau)\\
&p^{1}: u_{1}(\chi,\gamma,\tau)=-L_3^{-1}\left(\beta^{-1}v^{-1}L_3\left[Q_{0}(\chi,\gamma,\tau)\right]\right),\\
&p^{2}: u_{2}(\chi,\gamma,\tau)=-L_3^{-1}\left(\beta^{-1}v^{-1}L_3\left[Q_{1}(\chi,\gamma,\tau)\right]\right),\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&p^{n}: u_{n}(\chi,\gamma,\tau)=-L_3^{-1}\left(\beta^{-1}v^{-1}L_3\left[Q_{n-1}(\chi,\gamma,\tau)\right]\right),
\end{aligned}$$

Finally, we find the approximate analytical solution $u(\chi,\gamma,\tau)$ by truncated series

$$u(\chi,\gamma,\tau)=u_{0}(\chi,\gamma,\tau)+\sum_{n=1}^{\infty}u_{n}(\chi,\gamma,\tau), \tag{13}$$

Example 3.1: Consider the linear partial differential equation,

$$\beta \mathrm{u}_{\tau}(\chi,\gamma,\tau)=\mathrm{u}_{\gamma\gamma}(\chi,\gamma,\tau)+\mathrm{u}_{\chi\chi}(\chi,\gamma,\tau)+e^{\chi+\gamma} \tag{14}$$

With the initial conditions $\mathrm{u}(\chi,\gamma,0)=f(\chi,\gamma)=\mathrm{d}$, (d is constant)

Solution: From equation (12) we get the powers series of p as follow:

$$p^0 = u_0(\chi,\gamma,\tau) = d$$

$$p^1 = u_1(\chi,\gamma,\tau) = \frac{\tau}{\beta}e^{\chi+\gamma}$$

$$p^2 = u_2(\chi,\gamma,\tau) = \frac{\tau^2}{2\beta^2}e^{\chi+\gamma}$$

$$p^3 = u_3(\chi,\gamma,\tau) = \frac{\tau^3}{6\beta^3}e^{\chi+\gamma}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$p^n = u_n(\chi,\gamma,\tau) = \frac{\tau^n}{n!\beta^n}e^{\chi+\gamma}$$

Thus,

$$u(\chi,\gamma,\tau) = \sum_{n=0}^{\infty} u_n(\chi,\gamma,\tau) = d + \sum_{n=0}^{\infty} \frac{\tau^{n+1}}{(n+1)!\beta^{n+1}}e^{\chi+\gamma},$$

$$u(\chi,\gamma,\tau) = d + e^{\chi+\gamma}\left(e^{\frac{\tau}{\beta}} - 1\right)$$

3.2. The Convergence of the Solution TLTHPM for Linear PDEs

Lemma 3.2.1 If f be continues function then,

$$\frac{\partial}{\partial\tau}\int_0^{\tau} f(\tau - s)ds = f(\tau)$$

Proof

Suppose that $\int f(\chi)d\chi = F(\chi) + c$

Assume that

$$\chi = \tau - s \Rightarrow d\chi = -ds$$

$$\Rightarrow \frac{\partial}{\partial\tau}\int_0^{\tau} f(\tau - s)ds = -\frac{\partial}{\partial\tau}\int_{\tau}^{0} f(\chi)d\chi = \frac{\partial}{\partial\tau}\int_0^{\tau} f(\chi)d\chi$$

$$= \frac{\partial}{\partial\tau}\left[F(\tau) - F(0)\right] = \frac{\partial}{\partial\tau}F(\tau) - \frac{\partial}{\partial\tau}F(0) = f(\tau)$$

$$\therefore \frac{\partial}{\partial\tau}\int_0^{\tau} f(\tau - s)ds = f(\tau)$$

Lemma 3.2.2 Let L_3 is triple Laplace transform, then

$$\frac{\partial}{\partial\tau}\left[L_3^{-1}\left(L_3\left[f(\chi,\gamma,\tau)\right]\right)\right] = f(\chi,\gamma,\tau)$$

Proof: Let

$$\frac{\partial}{\partial\tau}\left[L_3^{-1}\left(L_3\left[f(\chi,\gamma,\tau)\right]\right)\right] = \frac{\partial}{\partial\tau}\left[L_3^{-1}\left(L_3[1]L_3\left[f(\chi,\gamma,\tau)\right]\right)\right]$$

$$= \frac{\partial}{\partial\tau}\left[L_3^{-1}\left(L_3\left[1 * f(\chi,\gamma,\tau)\right]\right)\right]$$

$$= \frac{\partial}{\partial\tau}\left[\int_0^{\tau} f(\chi,\gamma,\tau - s)ds\right] = f(\chi,\gamma,\tau)$$

Theorem 3.2.3 (Convergence Theorem)

f the series $u(\chi,\gamma,\tau)=u_0(\chi,\gamma,\tau)+\sum_{n=1}^{\infty}u_n(\chi,\gamma,\tau)$ is convergent, then the limit point converges to the exact solution of equation **(5)** which is calculated by TLTHPM

$$u_0(\chi,\gamma,\tau)+u_1(\chi,\gamma,\tau)=L_3^{-1}\left\{f(\chi,\gamma,\tau)-\beta^{-1}\nu^{-1}L_3\left[Q_0(\chi,\gamma,\tau)\right]\right\},$$

$$u_n(\chi,\gamma,\tau)=-\left\{L_3^{-1}\left(\beta^{-1}\nu^{-1}L_3\left[Q_{n-1}(\chi,\gamma,\tau)\right]\right)\right\},\ n>1$$

Proof: Let

$V(\chi,\gamma,\tau)=\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)$ converges to the limit point.

Now

$$\beta\frac{\partial}{\partial\tau}V(\chi,\gamma,\tau)=\beta\frac{\partial}{\partial\tau}\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)$$

$$=\beta\frac{\partial}{\partial\tau}\left[u_0(\chi,\gamma,\tau)+u_1(\chi,\gamma,\tau)+\sum_{n=2}^{\infty}u_n(\chi,\gamma,\tau)\right]$$

$$=\beta\frac{\partial}{\partial\tau}\left[L_3^{-1}\left\{f(\chi,\gamma)-\beta^{-1}\nu^{-1}L_3\left[Q\left[u_0(\chi,\gamma,\tau)\right]\right]\right\}-\sum_{n=2}^{\infty}L_3^{-1}\left(\beta^{-1}\nu^{-1}L_3\left[Q\left[u_{n-1}(\chi,\gamma,\tau)\right]\right]\right)\right]$$

$$=\beta\frac{\partial}{\partial\tau}f(\chi,\gamma)-Q\left[u_0(\chi,\gamma,\tau)\right]-\frac{\partial}{\partial\tau}\left(\sum_{n=2}^{\infty}L_3^{-1}\left(\nu^{-1}L_3\left[Q\left[u_n(\chi,\gamma,\tau)\right]\right]\right)\right)$$

$$=0-Q\left[u_0(\chi,\gamma,\tau)\right]-\sum_{n=0}^{\infty}\frac{\partial}{\partial\tau}\left[L_3^{-1}\left(\nu^{-1}L_3\left[Q\left[u_n(\chi,\gamma,\tau)\right]\right]\right)\right]$$

By lemma (3.2.2), we have:

$$\beta\frac{\partial}{\partial\tau}V(\chi,\gamma,\tau)=-Q\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)$$

Then, $V(\chi,\gamma,\tau)$ satisfies equation (5).

So, it is exact solution**.**

4- Triple Laplace Homotopy Perturbation Method for Nonlinear PDEs [18]

In this section, we will illustrate the basic idea of this method; we consider a general nonlinear nonhomogeneous partial differential equation

$$Ru(\chi,\gamma,\tau)+Qu(\chi,\gamma,\tau)+Nu(\chi,\gamma,\tau)=g(\chi,\gamma,\tau), \tag{15}$$

with the initial conditions

$$u(\chi,\gamma,0)=f(\chi,\gamma), \tag{16}$$

where $Ru(\chi,\gamma,\tau)=\frac{\partial u(\chi,\gamma,\tau)}{\partial\tau}$ is the partial derivative of the function $u(\chi,\gamma,\tau)$ of first order, $R\ \&Q$ is the linear differential operators, N represents the general nonlinear differential operator, and $g(x,y,t)$ is the source term.

Applying the triple Laplace transform on both sides of Eq. (15), we get:

$$L_3\left[Ru(\chi,\gamma,\tau)\right]+L_3\left[Qu(\chi,\gamma,\tau)\right]+L_3\left[Nu(\chi,\gamma,\tau)\right]=L_3\left[g(\chi,\gamma,\tau)\right], \tag{17}$$

Using the properties of triple Laplace transform, we obtain

$$\nu L_3\left[u(\chi,\gamma,\tau)\right]=f(\chi,\gamma)+L_3\left[g(\chi,\gamma,\tau)\right]-L_3\left[Qu(\chi,\gamma,\tau)+Nu(\chi,\gamma,\tau)\right], \tag{18}$$

we have:

$$L_3\left[u(\chi,\gamma,\tau)\right]=\nu^{-1}f(\chi,\gamma)+\nu^{-1}L_3\left[g(\chi,\gamma,\tau)\right]-\nu^{-1}L_3\left[Qu(\chi,\gamma,\tau)+Nu(\chi,\gamma,\tau)\right], \tag{19}$$

Operating the inverse triple Laplace transform on both sides of Eq. (19), to get:

$$u(\chi,\gamma,\tau)=f(\chi,\gamma)+L_3^{-1}\nu^{-1}\left(L_3\left[g(\chi,\gamma,\tau)\right]-L_3\left[Qu(\chi,\gamma,\tau)\right]-L_3\left[Nu(\chi,\gamma,\tau)\right]\right), \tag{20}$$

we apply the Homotopy perturbation method

$$u(\chi,\gamma,\tau)=\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau) \tag{21}$$

The nonlinear term can be decomposed in the following way:

$$N\left[u(\chi,\gamma,\tau)\right]=\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau), \tag{22}$$

Using the He's polynomial $H_n(u)$ given as follows:

$$H_{n}(u_0,u_1,u_2,\cdots u_n)=\frac{1}{\Gamma(n+1)}\frac{\partial^{n}}{\partial p^{n}}\left[N\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau)\right],n=0,1,2,\cdots \tag{23}$$

Substituting equations (12) and (13) into (11), to obtain,

$$\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau)=f(\chi,\gamma)+p\left\{L_{3}^{-1}\left(v^{-1}L_{3}\left[g(\chi,\gamma,\tau)-Q\sum_{n=0}^{\infty}p^{n}u_{n}(\chi,\gamma,\tau)+\sum_{n=0}^{\infty}p^{n}H_{n}(\chi,\gamma,\tau)\right]\right)\right\}, \tag{24}$$

This is the coupling of the triple Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficients of the like power of p, the following approximations are obtained:

$$p^{0}:u_{0}(\chi,\gamma,\tau)=f(\chi,\gamma)$$

$$p^{1}:u_{1}(\chi,\gamma,\tau)=L_{3}^{-1}\left(v^{-1}L_{3}\left[g(\chi,\gamma,\tau)-Q\left[u_{0}(\chi,\gamma,\tau)\right]-H_{0}(\chi,\gamma,\tau)\right]\right)$$

$$p^{2}:u_{2}(\chi,\gamma,\tau)=-L_{3}^{-1}\left(v^{-1}L_{3}\left\{Q\left[u_{1}(\chi,\gamma,\tau)\right]+H_{1}(\chi,\gamma,\tau)\right\}\right)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{n}:u_{n}(\chi,\gamma,\tau)=-L_{3}^{-1}\left(v^{-1}L_{3}\left\{Q\left[u_{n-1}(\chi,\gamma,\tau)\right]+H_{n-1}(\chi,\gamma,\tau)\right\}\right)$$

Finally, we approximate the analytical solution $u(x,y,t)$ by truncated series

$$u(\chi,\gamma,\tau)=u_{0}(\chi,\gamma,\tau)+\sum_{n=1}^{\infty}u_{n}(\chi,\gamma,\tau), \tag{25}$$

Example 4.1: Consider the nonlinear partial differential equation

$$\mathrm{u}_{\gamma}(\chi,\gamma,\tau)\mathrm{u}_{\tau}(\chi,\gamma,\tau)-\mathrm{u}_{\chi\chi}(\chi,\gamma,\tau)=\mathrm{u}(\chi,\gamma,\tau) \tag{26}$$

with the initial conditions

$$\mathrm{u}(0,\gamma,\tau) \;=\; \gamma\tau \;\;,\;\; \mathrm{u}_{\chi}(0,\gamma,\tau)=-1$$

Solution: From equation (24), we get the powers series of p as follow:

$$p^{0}=u_{0}(\chi,\gamma,\tau)=\gamma\tau-\chi$$

$$H_{0}u(\chi,\gamma,\tau)=\gamma\tau$$

$$\begin{aligned}p^{1}:u_{1}(\chi,\gamma,\tau)&=-L_{3}^{-1}\left(\frac{1}{\sigma^{2}}L_{3}\left[u_{0}(\chi,\gamma,\tau)-H_{0}u(\chi,\gamma,\tau)\right]\right)\\&=-L_{3}^{-1}\left(\frac{1}{\sigma^{2}}L_{3}\left[-\chi\right]\right)\\&=-L_{3}^{-1}\left(\frac{1}{\sigma^{2}}\left[\frac{-1}{\sigma^{2}\rho v}\right]\right)\end{aligned}$$

$$u_{1}(\chi,\gamma,\tau)=\frac{\chi^{3}}{3!}$$

$$H_1u(\chi,\gamma,\tau)=\frac{\partial}{\partial\gamma}u_0(\chi,\gamma,\tau)\frac{\partial}{\partial\tau}u_0(\chi,\gamma,\tau)+\frac{\partial}{\partial\gamma}u_1(\chi,\gamma,\tau)\frac{\partial}{\partial\tau}u_1(\chi,\gamma,\tau)$$

$$=\gamma\tau$$

$$p^2:u_2(\chi,\gamma,\tau)=-L_3^{-1}\left(\frac{1}{\sigma^2}L_3\left[u_1(\chi,\gamma,\tau)-H_1u(\chi,\gamma,\tau)\right]\right)$$

$$=-L_3^{-1}\left(\frac{1}{\sigma^2}L_3\left[\left(\frac{\chi^3}{3!}\right)-(\gamma\tau)\right]\right)$$

$$=-L_3^{-1}\left(\frac{1}{\sigma^2}\left[\frac{1}{\sigma^4\rho\nu}\right]-\left[\frac{1}{\sigma\rho^2\nu^2}\right]\right)$$

$$u_2(\chi,\gamma,\tau)=-\frac{\chi^5}{5!}+\frac{\chi^2}{2}\gamma\tau$$

The approximate series solution is,

$$u(\chi,\gamma,\tau)=\gamma\tau-\chi+\frac{\chi^3}{3!}-\frac{\chi^5}{5!}+\frac{1}{2}\chi^2\gamma\tau+\frac{\chi^7}{7!}\ldots$$

This can be written as:

$$u(\chi,\gamma,\tau)=\gamma\tau-\left[\chi-\frac{\chi^3}{3!}+\frac{\chi^5}{5!}-\ldots\right]$$

By using Taylor’s series, the closed form solution will be as follows

$$u(\chi,\gamma,\tau)=\gamma\tau-\sin\chi$$

4.2 The Convergence of the Solution TLTHPM for Nonlinear PDEs

Theorem 4.2.1 (Convergence Theorem)

Suppose that equation (25) converges, then we write the limit point as

$$V(\chi,\gamma,\tau)=\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)$$

$$\Rightarrow\frac{\partial}{\partial\tau}V(\chi,\gamma,\tau)=\frac{\partial}{\partial\tau}\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)$$

$$=\frac{\partial}{\partial\tau}\left[u_0(\chi,\gamma,\tau)+\sum_{n=1}^{\infty}u_n(\chi,\gamma,\tau)\right]$$

$$=\frac{\partial}{\partial\tau}\left[L_3^{-1}\{f(\chi,\gamma)\}-\sum_{n=1}^{\infty}L_3^{-1}\nu^{-1}\left\{L_3Q\left[u_{n-1}(\chi,\gamma,\tau)\right]+L_3\left[H_{n-1}(\chi,\gamma,\tau)\right]\right\}\right]$$

$$=\frac{\partial}{\partial\tau}\left[L_3^{-1}\{f(\chi,\gamma)\}-\sum_{n=0}^{\infty}L_3^{-1}\nu^{-1}\left\{L_3Q\left[u_n(\chi,\gamma,\tau)\right]\right\}-\sum_{n=0}^{\infty}L_3^{-1}\nu^{-1}\left\{L_3\left[H_n(\chi,\gamma,\tau)\right]\right\}\right],$$

$$=\frac{\partial}{\partial t}f(x,y)-\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}L_3^{-1}\delta^{-1}\left\{L_3Q\left[u_n(x,y,t)\right]\right\}\right]-\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}L_3^{-1}\delta^{-1}\left\{L_3\left[H_n(x,y,t)\right]\right\}\right]$$

$$\frac{\partial}{\partial\tau}V(\chi,\gamma,\tau)=\frac{\partial}{\partial\tau}f(\chi,\gamma)-\sum_{n=0}^{\infty}\frac{\partial}{\partial\tau}\left(L_3^{-1}\nu^{-1}\left\{L_3Q\left[u_n(\chi,\gamma,\tau)\right]\right\}\right)-\sum_{n=0}^{\infty}\frac{\partial}{\partial\tau}\left(L_3^{-1}\nu^{-1}\left\{L_3\left[H_n(\chi,\gamma,\tau)\right]\right\}\right),\qquad(27)$$

By lemma (3.2.2), equation (27) becomes.

$$\frac{\partial}{\partial\tau}V(\chi,\gamma,\tau)=0-\sum_{n=0}^{\infty}Q\left[u_n(\chi,\gamma,\tau)\right]-\sum_{n=0}^{\infty}H_n(\chi,\gamma,\tau)$$
$$=-Q\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)-N\sum_{n=0}^{\infty}u_n(\chi,\gamma,\tau)$$

6-Conclusion

In this work, we discussed the definition of the triple Laplace homotopy perturbation method, some important theorems and properties have been presented for this relatively new transformation to find the solutions for linear and nonlinear partial differential equations in three-dimensional space under the initial conditions. The triple Laplace homotopy perturbation method was studied to achieve the solutions. The convergence of the obtained solution to the exact solution by using the triple Laplace homotopy perturbation method is proved.

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