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Some types of compactness in topological spaces

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Abstract

This research explores the hierarchical structure among different generalizations of open sets and compactness within topological spaces. Specifically, it considers α-open, semi-open, pre-open, and regular open sets, along with their associated compactness concepts, including α- compactness, semi-compactness, strong compactness, and regular compactness. The study establishes several inclusion relations among these notions, supported by formal proofs and illustrative examples demonstrating that these implications are generally not reversible. Our results contribute to a deeper structural understanding of generalized topological properties and their interconnections, providing a clear reference for further research in general topology and related fields.

Keywords: topological spaces, α-open sets, compactness, α-compactness, semi-compactness, strong compactness, regular compactness, generalized open sets, hierarchy of compactness.

Introduction

Topological space theory is a cornerstone of modern mathematics, providing a comprehensive framework for studying the geometric and analytical properties of mathematical spaces in an abstract manner. Among the central concepts in this field is compactness, which serves as a powerful tool in analysis and topology due to its guarantees of the existence of limits and subsequential convergences in infinite spaces. This research aims to explore and enhance the understanding of different types of open sets in topological spaces, such as α-open sets, semi-open sets, pre-open sets, and regular open sets, as well as to study the associated compactness concepts, such as α-compactness, strong compactness, semi-compactness, and regular compactness. The primary objective is to examine the relationships and the hierarchical structure among these types of compactness and to present illustrative examples highlighting their mutual independence.

The importance of this study stems from the need for researchers and specialists in topology and mathematical analysis to gain a deeper understanding of the structural properties of compact spaces in their extended concepts, which are used in various fields such as functional analysis, differential geometry, and set theory.

These concepts also contribute to the development of more general and applicable theories in broader mathematical contexts. In this presentation, clear definitions of each type of open set and compactness will be provided, followed by theorems proving the relationships between them, while highlighting counterexamples that show these relationships are not necessarily reversible. The content will be organized in a clear academic style, with attention to logical order and precise mathematical reasoning, making the material easy to follow for the specialized reader.

Finally, we hope that this work will serve as a useful reference for students and researchers and contribute to enriching the Arabic mathematical library with accurate and organized scientific material in the field of general topology.

Definition 1.1. Let X be a non-empty set. A collection τ of subsets of X is called a **topology** on X if and only if τ satisfies the following axioms:

1. Both the empty set $\emptyset$ and the set X belong to τ.
2. The union of any collection (finite or infinite) of sets in τ also belongs to τ.
3. The intersection of any two sets in τ belongs to τ.

The members of τ are called τ-open sets, or simply open sets.
The pair (X, τ) is called a **topological space** (Lipschutz 1965) [8].

Example: Let X = {1,3,6,8} and τ = {Ø, X, {1}}

⇒ τ is a topology on X.
⇒ (X, τ) is a topological space.

1. Types of Open Sets in Topological Spaces

Definition 2.1. A subset A of a topological space X is called" α**-open set**" if and only if $A \subseteq \text{Int}(\text{cl}(\text{Int}(A)))$.

The family of all α-open sets is denoted by τ_α. [8]

Definition 2.2. A subset of a topological space is termed an **α-closed set** if its complement is an α-open set. The collection of all α-closed sets in X is denoted by αC(X). [8]

Example 1. Consider the set X = {1, 2, 3} equipped with the topology τ = {X, ∅, {1}, {2}, {1, 2}}.

In this topological space (X, τ), we observe that

⇒ τ_α = {∅, {1}, {2}, {1, 2}, X}

Definition 2.3. A subset S of the topological space (X, τ) is termed **semi-open** if it satisfies the condition $S \subseteq \text{Cl}(\text{Int}(S))$. The collection of all semi-open sets is denoted by SO(X). [4]

Example 2. Consider X = {1, 2, 3} be a topological space such that τ = {∅, {1}, {2, 3}, X}
⇒ SO(X) = {∅, {1}, {2, 3}, X}

Definition 2.4. Let (X, τ) be a topological space. A subset $S \subseteq X$ is called **pre-open** (p-open in brief) if $S \subseteq \text{Int}(\text{cl}(S))$.

The family comprising all pre-open subsets of X is represented by the notation PO(X). [4]

Example 3. Let X = {1, 2, 3} and τ = {∅, X, {1}, {2, 3}}

⇒ PO(X) = {∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, X}

Definition 2.5. A subset S of a topological space (X, τ) is said to be **quasi-open** if

$S \subset \mathrm{Cl}(\mathrm{Int}(S))$. The concept of quasi-open sets was introduced by Levine under a different terminology (see [1]).

Definition 2.6. A subset M of a topological space (X, τ) is called a **regular open set** if it satisfies the condition.

$$M = \mathrm{Int}(\mathrm{CL}(M)).$$

The family of all regular open sets of X is denoted by RO(X) [4].

Example 4. Let (R, τ) be a topological space on real numbers. Let $A = (-2, 5)$

$$\Rightarrow \quad \mathrm{CL}(A) = \mathrm{CL}\,((-2, 5)) = [-2, 5]$$

$$\mathrm{Int}(\mathrm{CL}(A)) = \mathrm{Int}\,([-2, 5]) = (-2, 5)$$

$$\Rightarrow A = \mathrm{Int}(\mathrm{CL}(A)) \Rightarrow A \text{ is regular-open set.}$$

3. Relation between types of open Sets in Topological Spaces

Theorem 3.1. Every open set in a topological space is α-open [4].

Proof: Let U be an open set in a topological space (X, τ).

To prove: $U \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$

Since U is open set ⇒ $\mathrm{Int}(U) = U$

Taking closure of both sides:

$$\mathrm{Cl}(\mathrm{Int}(U)) = \mathrm{Cl}(U)$$

Taking interior:

$$\mathrm{Int}\,(\mathrm{Cl}(\mathrm{Int}(U))) = \mathrm{Int}(\mathrm{Cl}(U))$$

Since U is open and $U \subseteq \mathrm{Cl}(U)$, we have:

$$U \subseteq \mathrm{Int}(\mathrm{Cl}(U))$$

(because U is the largest open set contained in Cl((U)) Therefore:

$$U \subseteq \mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$$

Then: U is α-open.

The converse need not be true. To show that, we can write the example:

Example 1. Let X = {a, b, c} and $\tau = \{\emptyset, \{a\}, X\}$.

Let $A = \{a, b\}$.

$$\mathrm{Int}(A) = \{a\}$$

$$\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Cl}(\{a\}) = \{a, b, c\}$$

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(\{a, b, c\}) = \{a, b, c\}$$

$$A = \{a, b\} \subseteq \{a, b, c\}$$

⇒ A is *an* α-open but not an open set.

Theorem 3.2. Every open set in a topological space is semi-open [16].

Proof: Let U be an open set in a topological space (X, τ).

(a) Since U is open, we have:

$$\text{Int}(U) = U$$

This follows from the definition of interior: for an open set, the interior equals the set itself

(b) Taking closure of both sides:

$$\text{Cl}(\text{Int}(U)) = \text{Cl}(U)$$

(c) By the definition of closure, for any set U :

$$U \subseteq \text{Cl}(U)$$

Substituting from step (b):

$$U \subseteq \text{Cl}(U) = \text{Cl}(\text{Int}(U))$$

Therefore: $U \subseteq \text{Cl}((\text{Int}(U)))$

which is exactly the condition for U to be semi-open.
The converse need not be true. To show that we can take the example:
Example 2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$.
Let $A = \{a, c\}$
⇒ A is not open.

$$\text{Int}(A) = \{a\}$$

$$\text{Cl}(\text{Int}(A)) = \text{Cl}(\{a\}) = X$$

$$A = \{a, c\} \subseteq X = \text{Cl}(\text{Int}(A))$$

Condition Satisfied.

$\Rightarrow A$ is semi-open.

Theorem 3.3. Let (X, τ) be a topological space. Then every α-open set in X is semi-open [16].
Proof: Let (X, τ) be a topological space. Assume: A is α-open.
We know this general fact in topology: For any set $B \subseteq X$:

$$\text{Int}(B) \subseteq B$$

Since A is α-open $\Rightarrow A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$

Now take $B = \text{Cl}(\text{Int}(A))$.

Then:

$$A \subseteq \text{Int}(\text{Cl}(\text{Int}(A))) \subseteq \text{Cl}(\text{Int}(A))$$

$\Rightarrow A \subseteq \text{Cl}(\text{Int}(A))$
$\Rightarrow A$ is a semi-open set.

The converse need not be true. To show that we can take the example:

Example 3. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Let $A = \{a, d\}$

$$\mathrm{Int}(A) = \mathrm{Int}(\{a, d\}) = \{a\}$$
$$\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Cl}(\{a\}) = \{a, d\}$$
$$A = \{a, d\} \subseteq \{a, d\}$$

⇒ Hence, A constitutes a semi-open set

$$\mathrm{Int}(A) = \{a\}$$
$$\mathrm{Cl}(\mathrm{Int}(A)) = \{a, d\}$$
$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(\{a, d\}) = \{a\}$$
$$A = \{a, d\} \nsubseteq \{a\}$$

Then A is semi-open but not α-open.

Theorem 3.4. In any topological space, every open set possesses the pre-open property [4].
Proof: Consider a topological space (X, τ). And let U denote an open set in X.

$$\Rightarrow U \in \tau$$

Since U is open $\Rightarrow \mathrm{Int}(U) = U$
(by definition of closure) $U \subseteq \mathrm{Cl}(U)$

The interior $\mathrm{Int}(\mathrm{Cl}(U))$ is the largest open set contained in $\mathrm{Cl}(U)$.

Since U is open and $U \subseteq \mathrm{Cl}(U)$, U is an open set contained in $\mathrm{Cl}(U)$.

Therefore, by the maximality of the interior:

$$U \subseteq \mathrm{Int}((\mathrm{Cl}(U)).$$

This establishes that U fulfills the pre-open condition.
The converse need not be true. To show that we can take the example:

Example 4. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$.

Let $A = \{a, b\}$.

since $A \notin \tau \Rightarrow A$ is not open set
$\mathrm{cl}(A) = \mathrm{cl}(\{a, b\}) = X$
$\mathrm{Int}(\mathrm{cl}(A)) = \mathrm{Int}(X) = X$
$\Rightarrow A = \{a, b\} \subseteq X = \mathrm{Int}(\mathrm{cl}(A))$
$\Rightarrow A \subseteq \mathrm{Int}(\mathrm{cl}(A))$
$\Rightarrow A$ is pre-open.

Theorem 3.5. Let (X, τ) be a topological space. Then every α-open set in X is pre-open [4].
Proof: Let $A \subseteq X$ be an α-open set. By definition of α-open sets, we have

$$A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$$

Since $\mathrm{Int}(A) \subseteq A$, we have:

$$\mathrm{Cl}(\mathrm{Int}(A)) \subseteq \mathrm{Cl}(A)$$

Taking the interior of both sides:

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \subseteq \mathrm{Int}(\mathrm{Cl}(A))$$

Therefore:

$$A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \subseteq \mathrm{Int}(\mathrm{Cl}(A))$$

So $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, which means A is pre-open.
The converse need not be true. To show that we can take the example:
Example 5. Let $X = \mathrm{R}$ with the standard topology.

Let $A = \mathrm{Q} \cap (0, 1)$

That is, $A = \{x \in \mathrm{R} \mid x \in \mathrm{Q},\ 0 < x < 1\}$

$$\mathrm{Int}(A) = \emptyset$$
$$\mathrm{Cl}((\mathrm{int}(A)) = \mathrm{Cl}\ (\emptyset) = \emptyset$$
$$\mathrm{Int}(\mathrm{Cl}(\mathrm{int}(A))) = \mathrm{Int}\ (\emptyset) = \emptyset$$
$$A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{int}(A)))$$
$$\Rightarrow A \not\subseteq \emptyset \Rightarrow A \text{ is not } \alpha\text{-open.}$$

since $\mathrm{Cl(A)} = [0,1]$
$\mathrm{int(Cl(A))} = \mathrm{int}\ ([0,1]) = (0,1)$
$\mathrm{A} \subseteq \mathrm{int}(\mathrm{Cl}(A))$
$\Rightarrow A \subseteq (0, 1)$
$\Rightarrow A$ is pre-open
Theorem 3.6. Every regular open set in a topological space (X, τ) is α-open set [4].
Proof: Let A be an arbitrary regular open set in the topological space (X, τ).
By definition, a set is regular open if it is equal to the interior of its closure, so:

$$A = \mathrm{Int}(\mathrm{Cl}(A))$$

Since the interior of any set is an open set in the topological space, it follows that

$\mathrm{Int}(\mathrm{Cl}(A))$ is an open set in (X, τ).
Therefore, A is an α-open set.

The converse need not be true. To show that we can take the example:
Example 6. Let the topological space be $X = \mathrm{R}$ with the standard topology.

Define the set A as the union of two distinct open intervals:

$$A = (0, 1) \cup (1, 2)$$

To demonstrate that A constitutes an open set, we observe that A represents the union of two open intervals. According to the definition of a topology, the union of any arbitrary collection of open sets inherently forms an open set. Consequently, a qualifies as an open set. A set is characterized as regular open if it fulfills the following condition:

$$A = \mathrm{Int}[\mathrm{Cl}(A)]$$

We will show this equality does not hold for our set A.

$$Cl(A) = [0, 2]$$

$$Int[Cl(A)] = (0, 2)$$

$$A = (0, 1) \cup (1, 2) \neq (0, 2)$$

Therefore, A is not regular open.

4. Types of Compactness in Topological Spaces

Definition 4.1. A Subset B of a topological space (X, τ) is said to be **Compact** if every open cover of B admits a finite subcover. [5]

- Let $B \subseteq (X, \tau)$, and let $\{ _i : i \in I \}$ be a family of open sets in X.

The family $\{O_i : i \in I \}$ is called an open cover of B if:

$$B \subseteq \cup_{i\in I} \ O_i$$

A finite subfamily $O_1, O_2, \ldots, O_m$ of $\{O_i\}$ is called a finite subcover of B if:

$$B \subseteq O_1 \cup O_2 \cup \ldots \cup O_m$$

Definition 4.2. A Subset A of a topological space (X, τ) is said to be an α**-compact space**

If every α-open cover of A has a finite subcover **[4]**.

- Let (X, τ) be a topological space and let $Y \subseteq X$.
- A family $\beta = \{U_i$:i$\in I\}$ of subsets of X

is called an α-open cover of Y if:

(i) Each U_i is an α-open set in X (i.e., $U_i \in T_\alpha$)

(ii) $The\ union\ covers\ Y\ (i.e. Y \ \subseteq \cup_{i\in I}\ Ui).$

Definition 4.3. A topological space (X, τ) is called a **regularly compact space** if every cover of X by regular open sets has a finite subcover. [4]

That is, for every family $\mu \subseteq RO(X)$ with $\cup \mu = X$, there exists a finite subfamily

$\{G_1, G_2, \ldots, G_m\}$ such that:

$$\cup_{i=1}^{m} Gi = X$$

Definition 4.4. A topological space (X, τ) is said to be **semi-compact** if every cover consisting of semi-open sets admits a finite subcover. [4]

Semi-open cover: A cover of X where all sets in the cover are semi-open.

Definition 4.5. A topological space. (X, τ) is termed **strong compact**

if every cover composed of pre- open sets possesses a finite subcover. [4]

Given a topological space (X, τ) and a subset $A \subset X$, a family

$$\{P_i, i \in I\}$$

of pre-open sets in τ is termed a pre-open cover of A

$$A \subset \cup_i \ pi$$

5. Relations Between Types of Compactness in Topological Spaces

Theorem 5.1. Every α-compact space is compact [8].

Proof: Let $\gamma = \{U_i\}_{i\in I}$ be an arbitrary open cover of X, i.e.:

$$\bigcup_{i\in I} Ui = X$$

Since every open set is also α-open, γ is an α-open cover of X. Since X is α-compact, there exists a finite subcollection $\{U_{i1}, U_{i2}, \ldots, U_{in}\}$ such that:

$U_{i1} \cup U_{i2} \cup \cdots \cup U_{in} = X$

Therefore, X is compact.
The converse need not be true, to show that we can take the example:
Example 7. Let $X = \mathrm{N}$ with the cofinite topology:

$$\tau = \{U \subseteq \mathrm{N} \mid \mathrm{N} \setminus U \text{ is finite}\} \cup \{\emptyset\}$$

In this topology, every open cover has a finite subcover. Therefore, (X, τ) is compact.

$$A \text{ is } \alpha\text{-open} \iff A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$$

If A is finite, then A is not α-open. All infinite subsets of X are α-open. Consider U is *an* α-open cover:

$$\gamma = \{U_n = X \setminus \{n\} \mid n \in \mathrm{N}\}$$

where $\bigcup_{n\in N} Un = X$.

However, for any finite subcollection $\{U_{n1}, \ldots, U_{nk}\}$:

$$U_{n1} \cup \cdots \cup U_{nk} = X \setminus \{n_1, \ldots, n_k\} \neq X$$

Therefore, (X, τ) is not α-compact.

Thus, (X, τ) is compact but not α-compact.
Theorem 5.2. Every semi-compact topological space is α-compact [4].
Proof: Let X be a semi-compact space and let $\{C_i\}_{i\in I}$ be a cover of X by α-open sets. By the lemma, every α-open set is semi-open. Therefore, $\{C_i\}_{i\in I}$ *is* also a semi-open cover of X. Since X is semi-compact, there exists a finite subcollection $\{C_{i1}, C_{i2}, \ldots, C_{in}\}$ such that:

$$C_{i1} \cup C_{i2} \cup \cdots \cup C_{in} = X$$

Hence, X is α-compact.
The converse need not to be true, to show that we can take the example:
Example 8. Let $X = \mathrm{R}$ with the usual topology. Every open set in R is α-open because:

$$U = \mathrm{int}(\mathrm{cl}(\mathrm{int}(U)))$$

α-open sets in R include all usual open sets.

Consider the collection:

$\gamma = \{(-n, n) \mid n \in \mathbb{N}\}$

This family U covers R but no finite subfamily covers all of R, because any finite union of $(-n, n)$ intervals misses points with $|x| > \max(n_k)$.

Thus, R is not semi-compact. However, under α-open coverings, some large α-open sets may cover with finitely many members. Therefore, it can be α-compact while failing to be semi-compact.

Theorem 5.3. Every strongly compact topological space possesses the property of α-compact [16].

Proof: Suppose X is a strongly compact space and consider an arbitrary cover $\{U_i\}_{i\in I}$ of X *consisting of* α-open stes. Given that every α-open set is necessarily pre-open, this α-open cover simultaneously constitutes a pre-open cover of
X, the pre-open cover admits a finite subcover. This consequently implies that the α-open cover of X possesses a finite subcover, establishing the *α-compactness* of X.

The converse need not be true, to show that we can take the example:

Example 9. Let $X = \{2n + 1 \mid n \in \mathbb{N}\}$ with the discrete topology

$\tau_\alpha = P(X)$.

Then (X, τ) is α-compact but not strongly compact.

Theorem 5.4. Every α-compact topological space is regularly compact [4].

Proof: Let $\{R_i\}_{i\in I}$ be a cover of X by regular open sets. Since every regular open set is α-open, $\{R_i\}_{i\in I}$ is an α-open cover of X. Since X is α-compact, every α-open cover has a finite subcover. Hence, there exists a finite subcollection $\{R_{i1}, R_{i2}, \ldots, R_{in}\}$ that covers X. Since all R_{ik} are regular open sets, the regular open cover $\{R_i\}_{i\in I}$ has a finite subcover. Therefore, X is regularly compact.

The converse need not to be true, to show that we can take the example:

Example 10. Let X= $\{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ be a topology on X.

Then the family of all regular open sets in X is given by

$$\Rightarrow R = \{\emptyset, \{a\}, \{b\}, X\}.$$

Since every regular open cover of X *must necessarily contain the set* X *itself, It* Follows that every regular open cover of X admits a finite subcover. Hence, X is a regular compact. Now, consider the following infinite α-open cover of X:

$\gamma = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, X\}$

Clearly, no finite subfamily of γ covers X.

Therefore, no finite subcover exists.

Theorem 5.5. Every compact topological space is regularly compact [5].

Proof: Let (X, τ) be a compact space. Let $\{U_i\}_{i\in I}$ be an arbitrary family of regular

Open sets covering X:

$$X = \bigcup_{i\in I} Ui$$

By definition, for each $i \in I$, $U_i = \mathrm{int}(\mathrm{cl}(U_i))$. Since every regular open set is an open set (because the interior of any set is always open), we have $U_i \in \tau$ for every $i \in I$. The collection $\{U_i\}_{i\in I}$ is now seen as an open cover of X. Since X is compact by assumption, this open cover must have a finite subcover. This means there exists a finite set of indices

$\{i_1, i_2, \ldots, i_n\} \subseteq I$ such that:

$$X = U_{i1} \cup U_{i2} \cup \cdots \cup U_{in}$$

We have shown that an arbitrary cover of X by regular open sets has a finite subcover. Therefore, by definition, X is regularly compact.
The converse need not to be true, to show that we can take the example:
Example 11. Let $X = \mathrm{N}$ (the set of natural numbers) with the cofinite topology.
Open sets: ∅ and all sets whose complements are finite.

- A set A is regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$

Let $U \neq \emptyset$, N be any open set in the cofinite topology.

$\mathrm{Cl}(U) = \mathrm{N}$ (since closure of any infinite set is N)

$$\mathrm{Int}(\mathrm{N}) = \mathrm{N}$$

$$\text{So } \mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{N} \neq U$$

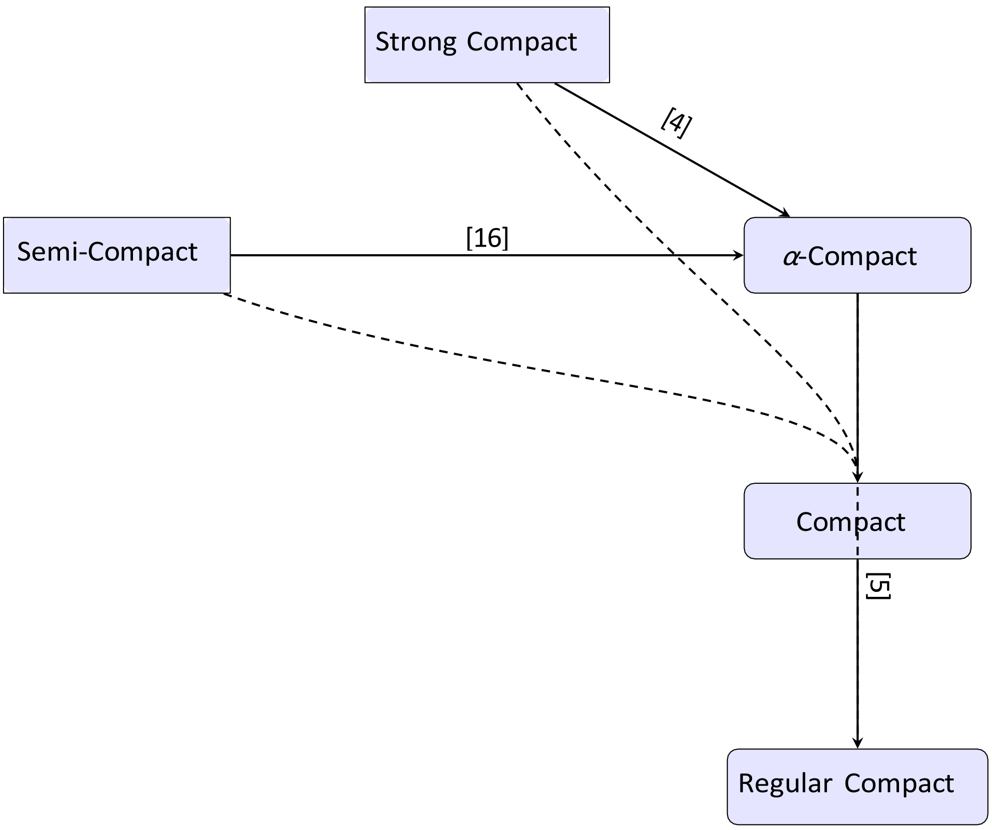
Consequently, the collection of regular" open sets consists exclusively of ∅ N. The definition of regular compactness stipulates that every cover composed of regular open sets admits a finite subcover. The unique regular open cover of N is the singleton {N}. This cover trivially possesses a finite subcover, namely {N} itself. Hence, the space (N, cofinite) satisfies regular compactness. Now consider the open cover defined by Un=N/ {n} for each n ∈ N. Each set Un is open since its complement {n} is finite

This cover has no finite subcover.

Any finite collection $\{U_{n1}, U_{n2}, \ldots, U_{nk}\}$ misses the number max $\{n_1, n_2, \ldots, n_k\}$.

Therefore, (N, cofinite) is not compact.

Results: From the theorems above, we can conclude the hierarchy of compactness notions



Note: Arrows indicate implications. References
In brackets correspond to theorem numbers.
Dashed arrows represent transitive implications
derived from the chain.

Summary of Relationships:

- α-Compact $\Longrightarrow$ Compact $\Longrightarrow$ Regular Compact
- Strong Compact $\Longrightarrow$ α-Compact $\Longrightarrow$ Compact $\Longrightarrow$ Regular Compact
- Semi-Compact $\Longrightarrow$ α-Compact $\Longrightarrow$ Compact $\Longrightarrow$ Regular Compac

Conclusions

This study investigated various forms of compactness and their associated concepts within the framework of topological spaces. The research provided a comprehensive clarification of the relationships between different types of compactness, emphasizing both their commonalities and distinctions through rigorous definitions and theoretical findings.

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