Journal of Education for Pure Science
Vol.16, No. 02 (2026)
DOI: https://doi.org/10.32792/jeps.v16i2.902

Website: jceps.utq.edu.iq Email: jceps@eps.utq.edu.iq

Hesitant Intuitionistic Fuzzy Two Absorbing Submodule

Rajaa Raihan Rasool [1*], Mohammed J. Mohammed [2], Ahmed H. Alwan [3]

[1]Department of Mathematics, College of Education for Pure Sciences, Thi-Qar University, Thi-Qar, Iraq.

[2]Department of Mathematics, College of Education, Al Ayen University, Thi-Qar, Iraq.

[3]Department of Mathematics, College of Education for Pure Sciences, Thi-Qar University, Thi-Qar, Iraq.

[*] Corresponding email: Rajaa_raihan_r@utq.edu.iq

Received 22/ 02 /2026, Accepted 22 / 05 /202X, Published 01/ 06 /2026

Abstract:

In this paper, we have defined the notion of hesitant intuitionistic fuzzy two absorbing sub-module, and present certain related results, including : level of A ($A_{H_{(\alpha,\beta)}}$) is two absorbing sub-module of R-module M, and we have defined the notion of hesitant intuitionistic F-prime sub-module, we also proved the relationship between hesitant intuitionistic fuzzy two absorbing sub-module and hesitant intuitionistic F-prime sub-module such that every hesitant intuitionistic F-prime sub-module is hesitant intuitionistic fuzzy two absorbing sub-module of R-module M **.**

Keywords: Hesitant intuitionistic fuzzy sub-module, hesitant intuitionistic fuzzy prime sub-module, hesitant intuitionistic fuzzy two-absorbing sub-module

1-Introduction.

Zadeh [1] introduced the fuzzy set, which is a mapping from a set 𝕏 into the closed interval L=[0,1]. S. Nanda defined the notion of the fuzzy module over fuzzy rings [2]. Subsequently, Atanassov [3] modified the concept of intuitionistic fuzzy sets, which expanded on the notion of fuzzy sets and described some of their properties.

The concept of a hesitant fuzzy set presented by Torra [4]. The notion of a two-absorbing sub-module was introduced by Ahmad and Fatemeh [5] as an extension of a prime submodule. Isaac and John introduced the intuitionistic fuzzy sub-module [6]. Rashid and Beg have defined a hesitant intuitionistic fuzzy set [7].

Hesitant intuitionistic fuzzy soft sets were introduced by Nazra, Syafruddin, and Wicaksono [8]. Khalaf and Hanoon introduced the two-absorbing fuzzy modules, two-absorbing fuzzy sub-modules, and some of their generalizations were presented [9]. Hesitant fuzzy modules were first proposed by Fadhil, Mohammed, and Hadi [10]. In this paper, we introduce a hesitant intuitionistic fuzzy two-absorbing sub-module of R-module M and give a number of associated results.

2.Preliminaries

In this section, we provide some definitions, and preliminary results that will be used in the next sections.

Definition 2.1[4]: A hesitant F-set (HFS) on a non-empty set $\mathbb{X}$ is a function $\hbar$ from $\mathbb{X}$ into subset of the interval L. Such that, $\hbar: \mathbb{X} \to \mathcal{P}(L)$. where E $= \{(e, \hbar_E(e)) : e \in \mathbb{X}\}$, and $\hbar_E(e)$ is a set of some values in L.

Definition 2.2[7]: A hesitant intuitionistic F-set (HIFS) on a non-empty set $\mathbb{X}$ is a set E = $\{(e, \grave{\mu}_{\hbar_E}(e), \grave{v}_{\hbar_E}(e)) / e \in \mathbb{X}\}$, where $\mu, v : \mathbb{X} \to \mathcal{P}(L)$. Such that $\grave{\mu}_{\hbar_E}(e)$ is a membership and $\grave{v}_{\hbar_E}(e)$ is a non-membership for $e \in \mathbb{X}$, which satisfy $\grave{\mu}_{\hbar_E}{}^{-}(e) + \grave{v}_{\hbar_E}{}^{+}(e) \leq 1$ and $\grave{\mu}_{\hbar_E}{}^{+}(e) + \grave{v}_{\hbar_E}{}^{-}(e) \leq 1$.

Definition 2.3[8]: Let E and F are two be HIFSs of the forms E= $\{(e, \grave{\mu}_{\hbar_E}(e), \grave{v}_{\hbar_E}(e)) / e \in \mathbb{X}\}$
$F = \{< e, \grave{\mu}_{\hbar_F}(e), \grave{v}_{\hbar_F}(e) > / e \in \mathbb{X}\}$ on $\mathbb{X}$, then:

i) $E \subset F$ *iff* $\grave{\mu}_{\hbar_E}(e) \subseteq \grave{\mu}_{\hbar_F}(e)$ *and* $\grave{v}_{\hbar_E}(e) \supseteq \grave{v}_{\hbar_F}(e)$ *for all* $e \in \mathbb{X}$.

ii) $E = F$ *iff* $E \subseteq F$ *and* $E \supseteq F$.

iii) $E^c = \{(e, \grave{v}_{\hbar_E}(e), \grave{\mu}_{\hbar_E}(e)) : e \in \mathbb{X}\}$.

iv) $E \cup F = \{(e, \grave{\mu}_{\hbar_E}(e) \cup \grave{\mu}_{\hbar_F}(e), \grave{v}_{\hbar_E}(e) \cap \grave{v}_{\hbar_F}(e)) : e \in \mathbb{X}\}$.

Where:

$= \cup_{y_1 \in \grave{\mu}_{\hbar_E}(e), y_2 \in \grave{\mu}_{\hbar_F}(e)} max\{y_1, y_2\}$, and $(\grave{\mu}_{\hbar_E} \cup \grave{\mu}_{\hbar_F})(e)$

$= \cup_{y_1 \in \grave{v}_{\hbar_E}(e), y_2 \in \grave{v}_{\hbar_F}(e)} min\{y_1, y_2\}.(\grave{v}_{\hbar_E} \cap \grave{v}_{\hbar_F})(e)$

iiv) $E \cap F = \{(x, \grave{\mu}_{\hbar_E}(e) \cap \grave{\mu}_{\hbar_F}(e), \grave{v}_{\hbar_E}(e) \cup \grave{v}_{\hbar_F}(e)) : e \in \mathbb{X}\}$.

Where:

$= \cup_{y_1 \in \grave{\mu}_{\hbar_E}(e), y_2 \in \grave{\mu}_{\hbar_F}(e)} min\{y_1, y_2\}$, and $(\grave{\mu}_{\hbar_E} \cap \grave{\mu}_{\hbar_F})(e)$

$(\grave{v}_{\hbar_E} \cup \grave{v}_{\hbar_F})(e) = \cup_{y_1 \in \grave{v}_{\hbar_E}(e), y_2 \in \grave{v}_{\hbar_F}(e)} max\{y_1, y_2\}$.

Definition 2.4[11]: Let $t_{(\tau,\rho)} : \mathbb{X} \to \mathcal{P}(L)$ be a HIFs of $\mathbb{X}$, such that $t \in \mathbb{X}$, $\tau, \rho \subseteq L$, define by

$$t_{(\tau,\rho)}(s) = \begin{cases} (\tau, \rho) & : \quad t = s \\ (\emptyset, [0,1]) & : \quad t \neq s \end{cases}$$

, $t_{(\tau,\rho)}$ is called hesitant intuitionistic F-point (HIFP) of $\mathbb{X}$.

an HIFP $t_{(\tau,\rho)}$ is called belong to a HIFs E of $\mathbb{X}$ and denoted by $t_{(\tau,\rho)} \subseteq$ E, if $\tau \subseteq \grave{\mu}_{\hbar_E}(t)$ and $\rho \supseteq \grave{v}_{\hbar_E}(t)$.

Definition 2.5[11]: Let A be a hesitant intuitionistic F-set in $\mathbb{X}$, and $\tau, \rho \subseteq [0,1]$, the set
$A_{H(\tau,\rho)} = \{w \in \mathbb{X} : \grave{\mu}_{\hbar_A}(w) \supseteq \tau, \grave{v}_{\hbar_A}(w) \subseteq \rho\}$ is called (τ, ρ) -level set.

Definition2.6 [11]: A subset E of a set $\mathbb{X}$ is called characteristic hesitant intuitionistic F-set, and defined to be the structure: $\hbar\chi_E(w) = (\mu_{\hbar\chi_E}(w), v_{\hbar\chi_E}(w) : w \in \mathbb{X})$, where:

$$\mu_{\hbar\chi_E}(w) = \begin{cases} [0,1] & \text{if } w \in E \\ \emptyset & \text{if } w \notin E \end{cases} \qquad v_{\hbar\chi_E}(w) = \begin{cases} \emptyset & \text{if } w \in E \\ [0,1] & \text{if } w \notin E. \end{cases}$$

Definition 2.7[10] : A hesitant F-set $\hbar$ of module M is called hesitant F-module (HFM) of M if for all $w, g \in M, s \in R$,

(i) $\hbar(w - g) \supseteq \hbar(w) \cap \hbar(g)$ $\qquad$ (ii) $\hbar(sw) \supseteq \hbar(w)$.

Definition 2.8[9]: A proper sub-module H of a module M is said to be a two-absorbing sub-module, if for s, t ∈ R, w ∈ M and stw ∈ H, then sw∈ H or tw∈ H or st∈ (H:M).

Definition2.9[6]: An intuitionistic F-set E = (μ_E, v_E) of a module M is said to be an intuitionistic F-sub-module (IFSM) if:

1) $\mu_E(0) = 1$, $v_E(0) = 0$,

2) $\mu_E(w+g) \geq \mu_E(w) \wedge \mu_E(g)$, $v_E(w+g) \leq v_E(w) \vee v_E(g)$ $\forall\, w, g \in$ M,

3) $\mu_E(sw) \geq \mu_E(w)$, $v_E(sw) \leq v_E(w)$, $\forall w \in$ M, $s \in$ R.

Definition2.10[5]: Let G be a proper sub-module of M, G is called two- absorbing sub-module of M, if for any $e, d \in$ R, w$\in$ M and edw$\in$ G, then ew$\in$ G or dw$\in$ G or $ed \in$ (G:M).

3. New Results on Hesitant Intuitionistic Fuzzy Two-Absorbing Sub-module

In the section, we introduced definition of hesitant intuitionistic fuzzy two-absorbing sub-module, and prove some results.

Definition 3.1: A hesitant intuitionistic F- set E = $(\grave{\mu}_{\hbar_E}, \grave{v}_{\hbar_E})$ of R-module M is called hesitant intuitionistic F-module (HIFM) if:

1) $\grave{\mu}_{\hbar_E}(\theta) = [0,1]$, $\grave{v}_{\hbar_E}(\theta) = \emptyset$, where θ is a zero element of M,

2) $\grave{\mu}_{\hbar_E}(w+g) \supseteq \grave{\mu}_{\hbar_E}(w) \cap \grave{\mu}_{\hbar_E}(g)$ and $\grave{v}_{\hbar_E}(w+g) \subseteq \grave{v}_{\hbar_E}(w) \cup \grave{v}_{\hbar_E}(g)$, $\forall\, w, g \in$ M,

3) $\grave{\mu}_{\hbar_E}(sw) \supseteq \grave{\mu}_{\hbar_E}(w)$ and $\grave{v}_{\hbar_E}(sw) \subseteq \grave{v}_{\hbar_E}(w)$, $\forall\, w \in$ M, $s \in$ R.

E is called a HIFM (M). That is, we mean that E is a hesitant intuitionistic F-sub-module (HIFSM) of some M.

Example3.2: Consider $M = Z_2$ as a Z – module. A HIFS E by:

$$\grave{\mu}_{\hbar_E}(w) = \begin{cases} [0,1] & if \quad w = \bar{0} \\ [0.3,0.5] & if \quad w = \bar{1} \end{cases} \qquad \grave{v}_{\hbar_E}(w) = \begin{cases} \emptyset & if \quad w = \bar{0} \\ [0.2,0.4] & if \quad w = \bar{1} \end{cases} \qquad \text{for all } w \in Z$$

Definition 3.3: Let E and H are two hesitant intuitionistic F-sub-modules. We define their sum E + H as follows:

E + H = $\{(\grave{\mu}_{\hbar_{E+H}}(w), \grave{v}_{\hbar_{E+H}}(w)) : w \in \text{M}\}$,

$= \cup\{\grave{\mu}_{\hbar_E}(j) \cap \grave{\mu}_{\hbar_H}(k) : j, k \in M, w = j + k\}$, and$\grave{\mu}_{\hbar_{E+H}}(w)$

$= \cap\{\grave{v}_{\hbar_E}(j) \cup \grave{v}_{\hbar_H}(k) : j, k \in M, w = j + k\}.\grave{v}_{\hbar_{E+H}}(w)$

Definition 3.4: Let E be a hesitant intuitionistic F-sub-module, for each $s \in$ R. We define sE as follows:

E = $\{(\grave{\mu}_{\hbar_{sE}}(w), \grave{v}_{\hbar_{sE}}(w)) : w \in \text{M}\}$, where for each $w \in$ M, s

$= \cup\{\grave{\mu}_{\hbar_E}(j) : j \in M, w = sj\}$, and$\grave{\mu}_{\hbar_{sE}}(w)$

$= \cap\{\grave{v}_{\hbar_E}(j) : j \in M, w = sj\}.\grave{v}_{\hbar_{sE}}(w)$

Definition3.5 : Let E be a hesitant intuitionistic F-sub-module, and $s_{(\alpha,\beta)} \in$ HIFP(R), where $\alpha, \beta \subseteq [0,1]$. For any n ∈ M

, and $$\mu'_{\hbar_{s_{(\alpha,\beta)}\cdot \text{E}}}(y) = \begin{cases} \cup\{\alpha \cap \grave{\mu}_{\hbar_E}(n) & if \quad y = sn,\ s \in R, n \in M\} \\ \emptyset & if \quad \text{other wise} \end{cases}$$

$$v_{\grave{k}_{s_{(\alpha,\beta)}\cdot E}}(y) = \begin{cases} \cap\{\beta \cup v_{\grave{k}_E}(n) & if \quad y = sn,\ s \in R, n \in M\} \\ \emptyset & if \quad \text{other wise}. \end{cases}$$

Proposition 3.6: Let E and H are two hesitant intuitionistic F-sub-modules, then:

1) $s(tE) = (st)E$ for any $s, t \in R$,

2) $s(E + H) = sE + sH$, for $s \in R$.

Definition 3.7: Let $r_{(\delta,\gamma)}$ be hesitant intuitionistic F-point of R, and $y_{(\sigma,\tau)}$, $x_{(\alpha,\beta)}$ be hesitant intuitionistic F-points of M, then:

(1) $r_{(\delta,\gamma)}x_{(\alpha,\beta)} = (rx)_{(\delta\cap\alpha,\gamma\cup\beta)}$,
(2) $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} = (x+y)_{(\alpha\cap\sigma,\beta\cup\tau)}$.

Proposition 3.8: Let E be a HIFSM of M, and $y_{(\sigma,\tau)}$, $x_{(\alpha,\beta)}$ be HIFPs of E, then :
1) $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} \subseteq E$. 2) $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq E$, for $r_{(\delta,\gamma)} \in$ HIFP(R).

Proof

1) Since $x_{(\alpha,\beta)} \subseteq E$, then $\mu_{\grave{k}_E}(x) \supseteq \alpha$, $v_{\grave{k}_E}(x) \subseteq \beta$ and $y_{(\sigma,\tau)} \subseteq$, then $\mu_{\grave{k}_E}(y) \supseteq \sigma$, $v_{\grave{k}_E}(y) \subseteq \tau$

Since $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} = (x+y)_{(\alpha\cap\sigma,\beta\cup\tau)}$ (by definition 3.7), and is a HIFSM of R-module M, then

$\mu_{\grave{k}_E}(x) \cap \mu_{\grave{k}_E}(y) \supseteq \alpha \cap \sigma$ and $v_{\grave{k}_E}(x+y) \subseteq v_{\grave{k}_E}(x) \cup v_{\grave{k}_E}(y) \subseteq \beta \cup \tau$, $\forall\, x, y \in M$.$\mu_{\grave{k}_E}(x+y) \supseteq$

Thus $\mu_{\grave{k}_E}(x+y) \supseteq \alpha \cap \sigma$ and $v_{\grave{k}_E}(x+y) \subseteq \beta \cup \tau$.

Implies $(x+y)_{(\alpha\cap\sigma,\beta\cup\tau)} \subseteq E$, so $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} \subseteq E$.

2) Since $r_{(\delta,\gamma)}x_{(\alpha,\beta)} = (rx)_{(\delta\cap\alpha,\gamma\cup\beta)}$ (by definition 3.7), and E is a HIFSM of R-module M, then

$\mu_{\grave{k}_E}(rx) \supseteq \mu_{\grave{k}_E}(x) \supseteq \alpha$ and $v_{\grave{k}_E}(rx) \subseteq v_{\grave{k}_E}(x) \subseteq \beta$, $\forall\, x \in M, r \in R$.

Implies $\mu_{\grave{k}_E}(rx) \supseteq \alpha$ and $v_{\grave{k}_E}(rx) \subseteq \beta$.

Since $\mu_{\grave{k}_E}(rx) \supseteq \alpha \supseteq \alpha \cap \sigma$ and $v_{\grave{k}_E}(rx) \subseteq \beta \subseteq \beta \cup \tau$.

Hence $\mu_{\grave{k}_E}(rx) \supseteq \alpha \cap \sigma$ and $v_{\grave{k}_E}(rx) \subseteq \beta \cup \tau$.

Therefore $(rx)_{(\delta\cap\alpha,\gamma\cup\beta)} \subseteq E$.

Implies $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq E$.

Definition 3.9: A hesitant intuitionistic F-sub-module E is called a hesitant intuitionistic F-two absorbing sub-module (in short, T-ABSO-HIFSM) of M, if for each $s_{(\sigma,\vartheta)}, r_{(\delta,\gamma)} \in$ HIFP(R), $x_{(\alpha,\beta)} \in$ HIFP(M) $\{s, r \in R, x \in M . \sigma, \vartheta, \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$, such that $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq$ E implies that $r_{(\delta,\gamma)}\, x_{(\alpha,\beta)} \subseteq$ E or $s_{(\sigma,\vartheta)}x_{(\alpha,\beta)} \subseteq$ E or $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq (E{:}\, \chi_{\grave{k}_M})$, where:

$$(E{:}\, \chi_{\grave{k}_M}) = \cup\{D | D \subseteq HIFSM(R) \mid D \cdot \chi_{\grave{k}_M} \subseteq E\}$$

$$= \cup\{r_{(\varepsilon,\rho)}{:}\, r \in R,\ \varepsilon, \rho \subseteq [0,1] \mid r_{(\varepsilon,\rho)}\, \chi_{\grave{k}_M} \subseteq E\}.$$

Theorem 3.10: Let E, H are two HIFSM, if E is T-ABSO-HIFSM, then $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}$H $\subseteq$ E, for each $s_{(\sigma,\vartheta)}, r_{(\delta,\gamma)} \in$ HIFP(R) implies that $r_{(\delta,\gamma)}$ H $\subseteq$ E or $s_{(\sigma,\vartheta)}$H $\subseteq$E or $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq (E{:}\, \chi_{\grave{k}_M})$.

Proof: Let E be is T-ABSO-HIFSM, and $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}H \subseteq E$. Assume that $r_{(\delta,\gamma)}\, H \nsubseteq E$ and $s_{(\sigma,\vartheta)}H \nsubseteq E$ and $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \nsubseteq (E:\chi_{k_M})$. Then there exist $x_{(\alpha,\beta)}, y_{(\rho,\tau)} \subseteq H$, such that $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \nsubseteq E$ and $s_{(\sigma,\vartheta)}y_{(\rho,\tau)} \nsubseteq E$.

Since $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}H \subseteq E$ and $x_{(\alpha,\beta)} \subseteq H$, implies that $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq E$.
Since $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \nsubseteq (E:\chi_{k_M})$ and $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \nsubseteq E$ and E be is T-ABSO-HIFSM. We have $s_{(\sigma,\vartheta)}x_{(\alpha,\beta)} \subseteq E$.
Also since $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}H \subseteq E$ and $y_{(\rho,\tau)} \subseteq H$, implies that $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}y_{(\rho,\tau)} \subseteq E$.
Since $s_{(\sigma,\vartheta)}y_{(\rho,\tau)} \nsubseteq E$ and $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \nsubseteq (E:\chi_{k_M})$, and E be is T-ABSO-HIFSM of M. We have $r_{(\delta,\gamma)}y_{(\rho,\tau)} \subseteq E$.
Now, since $x_{(\alpha,\beta)}, y_{(\rho,\tau)} \subseteq H$, then $x_{(\alpha,\beta)} + y_{(\rho,\tau)} \subseteq H$ [by Proposition 3.8].
Implies $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)}(x_{(\alpha,\beta)} + y_{(\rho,\tau)}) \subseteq E$,
Since $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \nsubseteq (E:\chi_{k_M})$we have $r_{(\delta,\gamma)}(x_{(\alpha,\beta)} + y_{(\rho,\tau)}) \subseteq E$ or $s_{(\sigma,\vartheta)}(x_{(\alpha,\beta)} + y_{(\rho,\tau)}) \subseteq E$.
If $r_{(\delta,\gamma)}(x_{(\alpha,\beta)} + y_{(\rho,\tau)}) \subseteq E$ then $r_{(\delta,\gamma)}x_{(\alpha,\beta)} + r_{(\delta,\gamma)}y_{(\rho,\tau)} \subseteq E$.
Since $r_{(\delta,\gamma)}y_{(\rho,\tau)} \subseteq E$ we get $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq E$, this is a contradiction.
If $s_{(\sigma,\vartheta)}(x_{(\alpha,\beta)} + y_{(\rho,\tau)}) \subseteq E$ then $s_{(\sigma,\vartheta)}x_{(\alpha,\beta)} + s_{(\sigma,\vartheta)}y_{(\rho,\tau)} \subseteq E$.
Since $s_{(\sigma,\vartheta)}x_{(\alpha,\beta)} \subseteq E$ we get $s_{(\sigma,\vartheta)}y_{(\rho,\tau)} \subseteq E$, this is a contradiction.
Thus either $r_{(\delta,\gamma)}\, H \subseteq E$ or $s_{(\sigma,\vartheta)}H \subseteq E$ or $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq (E:\chi_{k_M})$.

Theorem 3.11: Let A be a T-ABSO-HIFSM, then $A_{H_{(\alpha,\beta)}}$ is T-ABSO-SM of M .

Proof : Let $s, r \in R, x \in M$ such that $srx \in A_{H_{(\alpha,\beta)}}$.
Since $srx \in A_{H_{(\alpha,\beta)}}$ implies $\mu_{\grave{k}_A}(srx) \supseteq \alpha$ and $v_{\grave{k}_A}(srx) \subseteq \beta$ [By definition 2.4].
So, by definition (2.5) we get $(srx)_{(\alpha,\beta)} \subseteq A$ implies $s_{(\alpha,\beta)}r_{(\alpha,\beta)}x_{(\alpha,\beta)} \subseteq A$ [by definition 3. 7].
Since A be is a T-ABSO-HIFSM of M.
Thus, $r_{(\alpha,\beta)}\, x_{(\alpha,\beta)} \subseteq A$ or $s_{(\alpha,\beta)}\, x_{(\alpha,\beta)} \subseteq A$ or $s_{(\alpha,\beta)}r_{(\alpha,\beta)} \subseteq (A:\chi_{k_M})$.
If $r_{(\alpha,\beta)}\, x_{(\alpha,\beta)} \subseteq A$, then $(rx)_{(\alpha,\beta)} \subseteq A$ [by definition 3.7].
Implies $\mu_{\grave{k}_A}(rx) \supseteq \alpha$ and $v_{\grave{k}_A}(rx) \subseteq \beta$ [by definition 2. 4].

Then $rx \in A_{H_{(\alpha,\beta)}}$ [by definition 2.5].

If $s_{(\alpha,\beta)}\, x_{(\alpha,\beta)} \subseteq A$, then $(sx)_{(\alpha,\beta)} \subseteq A$.

Implies $\mu_{\grave{k}_A}(sx) \supseteq \alpha$ and $v_{\grave{k}_A}(sx) \subseteq \beta$.

Thus $sx \in A_{H_{(\alpha,\beta)}}$.

If $s_{(\alpha,\beta)}r_{(\alpha,\beta)}\chi_{h_M} \subseteq A$, so, $\mu_{k_{(sr)'_{(\alpha,\beta)}\cdot\chi_{k_M}}}(srm) \subseteq \mu_{\grave{k}_A}(srm)$

Since $\mu_{k_{(sr)'_{(\alpha,\beta)}\cdot\chi_{k_M}}}(y) = \cup\left\{\alpha \cap \mu_{\chi_{k_M}}(m): y = srm, \quad s, r \in R, m \in M\right\}$

$= \cup \{\alpha \cap [0,1]\} = \alpha$ [by definition 3.5].
Implies $\mu_{\grave{k}_A}(srm) \supseteq \alpha$.
Similarly,
$v_{k_{(sr)'_{(\alpha,\beta)}\cdot\chi_{k_M}}}(y) = \cap\left\{\beta \cup v_{\chi_{k_M}}(m): y = srm, \quad s, r \in R, m \in M\right\}$

$= \cap \{\beta \cup \emptyset\} = \beta$ [by definition 3.5] .
Implies $v_{\grave{k}_A}(srm) \subseteq \beta$.
Thus $srm \in A_{H_{(\alpha,\beta)}}$ implies $srM \subseteq A_{H_{(\alpha,\beta)}}$, for all $m \in M$.
Then $A_{H_{(\alpha,\beta)}}$ is T-ABSO-SM of M.

Theorem 3.12: Let A be a T-ABSO-HIFSM of B. If $A_{H(\alpha,\beta)} \neq B_{H(\alpha,\beta)}$, $\alpha, \beta \subseteq [0,1]$. Then $A_{H(\alpha,\beta)}$ is T-ABSO-SM of $B_{H(\alpha,\beta)}$.

Proof: Let $A_{H(\alpha,\beta)} \neq B_{H(\alpha,\beta)}$ and $srm \in A_{H(\alpha,\beta)}$, for some $s, r \in R$ and $m \in M$,

since $srm \in A_{H(\alpha,\beta)}$ implies $\grave{\mu}_{\hbar_A}(srm) \supseteq \alpha$ and $\grave{v}_{\hbar_A}(srm) \subseteq \beta$ [by definition 2.5].

Then $(srm)_{(\alpha,\beta)} = s_{(\alpha,\beta)} r_{(\alpha,\beta)} m_{(\alpha,\beta)} \subseteq A$ [by definition 3.7] .

Since A is a T-ABSO-HIFSM of B, then

either $s_{(\alpha,\beta)}\, m_{(\alpha,\beta)} \subseteq A$ or $r_{(\alpha,\beta)}\, m_{(\alpha,\beta)} \subseteq A$ or $s_{(\alpha,\beta)} r_{(\alpha,\beta)} \subseteq (A{:}\, \chi_{\hbar_B})$.

Case (i): If $s_{(\alpha,\beta)}\, m_{(\alpha,\beta)} \subseteq A$ then, $(sm)_{(\alpha,\beta)} \subseteq A$ [by definition 3.7].

Implies that $\grave{\mu}_{\hbar_A}(sm) \supseteq \alpha$ and $\grave{v}_{\hbar_A}(sm) \subseteq \beta$ [by definition 2.4]

So, $sm \in A_{H(\alpha,\beta)}$.

Case (ii): If $r_{(\alpha,\beta)}\, m_{(\alpha,\beta)} \subseteq A$, then $(rm)_{(\alpha,\beta)} \subseteq A$ [by definition 3.7].

Implies that $\grave{\mu}_{\hbar_A}(rm) \supseteq \alpha$ and $\grave{v}_{\hbar_A}(rm) \subseteq \beta$ [by definition 2.4].

So, $rm \in A_{H(\alpha,\beta)}$.

Case (iii):If $s_{(\alpha,\beta)} r_{(\alpha,\beta)} \subseteq (A{:}\, \chi_{\hbar_B})$, then for any $y \in srB_{H(\alpha,\beta)}$, $y = srx$, for some $x \in B_{H(\alpha,\beta)}$.

So, $\grave{\mu}_{\hbar_B}(x) \supseteq \alpha$ and $\grave{v}_{\hbar_B}(x) \subseteq \beta$ [by definition 2.4].

Now,

$$\alpha = \alpha \cap \grave{\mu}_{\hbar_B}(x) \subseteq \cup \{\alpha \cap \grave{\mu}_{\hbar_B}(z) : y = srz\} = \mu_{\hbar_{(sr)'_{(\alpha,\beta)}B}}(y) \subseteq \grave{\mu}_{\hbar_A}(y) \text{ [by definition 3.5].}$$

Similarly,

$$\beta = \beta \cup \grave{v}_{\hbar_B}(x) \supseteq \cap \{\beta \cup \grave{v}_{\hbar_B}(z) : y = srz\} = v_{\hbar_{(sr)'_{(\alpha,\beta)}B}}(y) \supseteq \grave{v}_{\hbar_A}(y) \text{ [by definition 3.5].}$$

Implies $\grave{\mu}_{\hbar_A}(y) \supseteq \alpha$ and $\grave{v}_{\hbar_A}(y) \subseteq \beta$.

Then $y \in A_{H(\alpha,\beta)}$, $srB_{H(\alpha,\beta)} \subseteq A_{H(\alpha,\beta)}$.

Hence $A_{H(\alpha,\beta)}$ is T-ABSO-SM of $B_{H(\alpha,\beta)}$.

Definition 3.13: Let E be a hesitant intuitionistic F-sub-module. Then E is called an hesitant intuitionistic F-prime sub-module (P-HIFSM(M), in short) of M, if for each $r_{(\delta,\gamma)} \in HIFP(R)$, $x_{(\alpha,\beta)} \in HIFP(M)$ $\{r \in R, x \in M . \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$, where $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$, then either $x_{(\alpha,\beta)} \subseteq E$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq E$.

Proposition 3.14: If E, H are two P-HIFSM of M, then $E \cap H$ be a T-ABSO-HIFSM

Proof : let $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E \cap H$ for each $s_{(\sigma,\vartheta)}, r_{(\delta,\gamma)} \in HIFP(R)$, $x_{(\alpha,\beta)} \in HIFP(M)\{s, r \in R, x \in M . \sigma, \vartheta, \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$.

Suppose that $r_{(\delta,\gamma)}\, x_{(\alpha,\beta)} \nsubseteq E \cap H$ and $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \nsubseteq E \cap H$.

Implies $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \nsubseteq E$ and $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \nsubseteq E$ or $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \nsubseteq H$ and $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \nsubseteq H$.

Since $s_{(\sigma,\vartheta)} r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E \cap H$ implies $s_{(\sigma,\vartheta)} r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$ and $s_{(\sigma,\vartheta)} r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq H$.

Since $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \nsubseteq E$, $s_{(\sigma,\vartheta)} r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$, and E is a P-HIFSM(M), implies $r_{(\delta,\gamma)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right)$.

Since $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \nsubseteq H$, $s_{(\sigma,\vartheta)} r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq H$, and H is a P-HIFSM(M), implies $s_{(\sigma,\vartheta)} \subseteq \left(H{:}\,\chi_{\mathscr{k}_M}\right)$.

Then $s_{(\sigma,\vartheta)} r_{(\delta,\gamma)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right) \cap \left(H{:}\,\chi_{\mathscr{k}_M}\right) = \left(E \cap H{:}\,\chi_{\mathscr{k}_M}\right)$.

then $E \cap H$ is T-ABSO-HIFSM of M.

Proposition 3.15: Every hesitant intuitionistic F-prime sub-module is T-ABSO-HIFSM of M.

Proof: Let E be a P-HIFSM(M). And let $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$ for each $s_{(\sigma,\vartheta)}, r_{(\delta,\gamma)} \in$ HIFP(R), $x_{(\alpha,\beta)} \in$ HIFP(M)$\{s, r \in R, x \in M\, .\, \sigma, \vartheta, \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$.

Let $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} = y_{(\varepsilon,\tau)}$ implies $y_{(\varepsilon,\tau)} x_{(\alpha,\beta)} \subseteq E$.

Since E be a P-HIFSM(M), so, $y_{(\varepsilon,\tau)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right)$ or $x_{(\alpha,\beta)} \subseteq E$.

Therefore $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right)$ or $x_{(\alpha,\beta)} \subseteq E$.

Since $x_{(\alpha,\beta)} \subseteq E$ and $r_{(\delta,\gamma)} \in$ HIFP(R), then $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$ [by proposition 3.8].

Thus $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right)$ or $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$.

Since $x_{(\alpha,\beta)} \subseteq E$ and $s_{(\sigma,\vartheta)} \in$ HIFP(R), then $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \subseteq E$ [by proposition 3.8].

Therefore $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right)$ or $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \subseteq E$.

Then $s_{(\sigma,\vartheta)}\, r_{(\delta,\gamma)} \subseteq \left(E{:}\,\chi_{\mathscr{k}_M}\right)$ or $s_{(\sigma,\vartheta)} x_{(\alpha,\beta)} \subseteq E$ or $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq E$.

Implies that E is T-ABSO-HIFSM of M.

Results

The obtained results introduced the concept of hesitant intuitionistic fuzzy two absorbing submodule and established several fundamental properties related to this structure. Moreover, the relationships between the proposed concept and existing notions such as fuzzy two absorbing submodule and intuitionistic fuzzy submodule were investigated. The study also showed that the new concept represents a generalization of previous fuzzy algebraic structures.

Conclusion

In conclusion, we study the main definitions and theorems related to hesitant intuitionistic fuzzy T-ABSO sub-module and we have obtained new results, which is the relationship between hesitant intuitionistic fuzzy T-ABSO sub-module and hesitant intuitionistic F-prime sub-module such that every hesitant intuitionistic F-prime sub-module is hesitant intuitionistic fuzzy T-ABSO sub-module of R-module M.

References

[1] **L. Zadeh**, “Fuzzy Sets,” Information and Control, vol. 8, 1965, pp. 338–353.

[2] **S. Nanda**, “Fuzzy Modules over Fuzzy Rings,” Bull. Cal. Math. Soc., vol. 81, 1985, pp. 197–200.

[3] **K. Atanassov**, “Intuitionistic Fuzzy Sets,” Fuzzy Sets and Systems, vol. 20, 1986, pp. 86–96.

[4] **V. Torra**, “Hesitant Fuzzy Sets,” International Journal of Intelligent Systems, vol. 25, 2010, pp. 529–539.

[5] **A. Y. Darani and F. Soheilnia**, “2-Absorbing and Weakly 2-Absorbing Submodules,” Thai Journal of Mathematics, vol. 9, 2010, pp. 577–584.

[6] **P. Isaac and P.P. John**, "On Intuitionistic Fuzzy Submodules of a Module," International Journal of Mathematical Sciences and Applications, vol. 1, no. 3, pp. 1447–1454, 2011.

[7] **T. Rashid and I. Beg**, "Hesitant Intuitionistic Fuzzy Set," International Journal of Fuzzy Logic and Intelligent Systems, vol. 14, no. 3, pp. 181–187, 2014.

[8] **A. Nazra and G. Wicaksono**, "Hesitant intuitionistic fuzzy soft sets," Series, vol. 890, 2017.

[9] **W. Hadi and H.Y. Khalaf**, "T-ABSO Fuzzy Submodules and T-ABSO Fuzzy Modules and Some Their Generalizations," M.Sc. Thesis, University of Baghdad, 2019.

[10] **A. Fadhil, M. J. Mohammad, and R. Hadi**, "On Hesitant Fuzzy Modules," M.Sc. Thesis, University of Thi-Qar, 2021.

[11] **R. Raihan and M. J. Mohammad**, "New Results Related with Hesitant Intuitionistic Fuzzy Ideals of a Ring," M.Sc. Thesis, University of Thi-Qar, 2022.