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Properties of Fractional Calculus Operators and Results of Their Applications to The Multiplier Transformations Operator Within a Different Class of Multivalent Functions

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Abstract:

This research aims to present and study some of the basic and important properties of fractional calculus by examining their applications, which involve the multiplier transformation and other generalized operators. This is done in light of defining a new class of multivalent functions and distinguishing them by multiplier transformations operator. We have studied and proven the most important distortion or deformation inequalities, including those involving fractional operators. At the forefront of these results is the determination of the coefficients. In fact, we have already explained most of the previous studies related to the subject. In conclusion, the article provides new approaches and results to studies presenSted in previous research on the topic.

Keywords: Multivalent functions, Fractional calculus operator, Multiplier transformation, Unit disc.

1-Introduction

Initially, we first took the class $\mathfrak{S}(\wp)$ as the basic class for this research article, whose elements are multivalent analytic functions within the unit disc $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$, which we express in the following form:

$$f(z) = z^{\wp} - \sum_{k=1}^{\infty} a_{\wp+k} z^{\wp+k}. \tag{1}$$

When $\wp$ is a natural number, and z is naturally in the open unit disc $\mathcal{U}$. In fact, we have abbreviated our choice of this class $\mathfrak{S}(\wp)$ because it is, in fact, a subclass of another class with the same properties, with the difference that the coefficients are not necessarily negative.

The multiplier transformations operator $\mathfrak{A}_{\wp}^{\nu}(\varsigma, \eta, \varrho)$, presented in reference [1], caught our attention while reviewing previous research on topic to gain as much insight into previous studies and results. It is defined in the following form:

$$\mathfrak{A}_{p}^{0}(\varsigma,\eta,\varrho)f(z)=f(z),$$

$$(p+\varrho)\mathfrak{A}_{p}^{1}(\varsigma,\eta,\varrho)f(z)=\varsigma\eta z^{2}f''(z)+\big(\varsigma-\eta+(1-p)\varsigma\eta\big)zf'(z)+\big(p(1-\varsigma+\eta)+\varrho\big)f(z),$$

$$\begin{aligned}(p+\varrho)\mathfrak{A}_{p}^{2}(\varsigma,\eta,\varrho)f(z)\\ &=\varsigma\eta z^{2}\big(\mathfrak{A}_{p}^{1}(\varsigma,\eta,\varrho)f(z)\big)''+(\varsigma-\eta+(1-p)\varsigma\eta)z\big(\mathfrak{A}_{p}^{1}(\varsigma,\eta,\varrho)f(z)\big)'\\ &+(p(1-\varsigma+\eta)+\varrho)\mathfrak{A}_{p}^{1}(\varsigma,\eta,\varrho)f(z),\end{aligned}$$

$$\mathfrak{A}_{p}^{\nu_1}(\varsigma,\eta,\varrho)\big(\mathfrak{A}_{p}^{\nu_2}(\varsigma,\eta,\varrho)f(z)\big)=\mathfrak{A}_{p}^{\nu_2}(\varsigma,\eta,\varrho)\big(\mathfrak{A}_{p}^{\nu_1}(\varsigma,\eta,\varrho)f(z)\big),$$

provided $\varsigma\geq\eta\geq 0,\nu,\varrho\geq 0$ are all positive real numbers, p is a natural number, and z is naturally in the open unit disc $\mathcal{U}$,

then for each element of the basic class $\mathfrak{S}(p)$,which is written in the form (1), the following relation is true:

$$\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)=z^{p}-\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}a_{p+k}z^{p+k}\,. \tag{2}$$

If we look closely at this operator, or rather examine it closely, we notice that it reduces to operators studies in the previous studies on the topic by taking the parameters to specific values. Therefore, it is nothing more than a generalization of operators that have been proposed in previous research articles below we mention some of these operator what caught our attention, for example, when $p=1,\varrho=0$, it is exactly the operator that was introduced in the reference [2], also in the case of $p=1,\eta=0$, it is the same operator in reference [3]. It is reduced to the operator studied in the reference, [4], if we assume the parameters $\varsigma=1,\eta=0$.

Let's not forget that the multiplier transformations operator has been used effectively by researchers in recent years. For example [5-10]. In our opinion, the basis of this research article is our definition of the following class, which is essentially a subclass of the fundamental class $\mathfrak{S}(p)$, in which we used the multiplier transformations operator $\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)$ as follows:

Definition 1.1. The function f, which is an element of the class $\mathfrak{S}(p)$ and whose formula is (1), belongs to the special class $\mathfrak{N}_{p}^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$ if the following condition is met:

$$\mathcal{R}e\left[e^{i\vartheta}\left\{\frac{(1-\tau)\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^{p}}+\frac{\tau\left(\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}}-\xi\right\}\right]>0, \tag{3}$$

this is only when $\tau\geq 0,\ \xi<1,\vartheta\in\mathbb{R}$ and $\varsigma\geq\eta\geq 0,\nu,\varrho$ are all positive real numbers, p is a natural number, and z is naturally in the open unit disc $\mathcal{U}$.

In the following sections, we present the most important results we have established for this different class $\mathfrak{N}_{p}^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$, which include finding the bounds of the coefficients and the inequalities of distortion that contain the fractional calculus operator, the multiplier transforms operator, and other generalized operators. The results we obtained represent new approaches to previously known results. It is worth noting that the fractional calculus operators have gained widespread use by researchers in recent years due to their interesting applications. Here, we would like to point out some of them, such as [11- 16].

An important point, from our point of view, is to clarify the fractional calculus operators $\mathfrak{D}_{z}^{\kappa},\mathfrak{D}_{z}^{-\kappa}$ and $\mathfrak{D}^{m+\kappa}$ for a function defined over a simple connected domain with an origin, as presented in [14], since I will use them in the next sections as following:

$$\mathfrak{D}_z^{\kappa} f(z) = \frac{1}{\Gamma(1-\kappa)} \int_0^z \frac{f(t)}{(z-t)^{\kappa}} dt, \qquad \kappa > 0, \tag{4}$$

$$\mathfrak{D}_z^{-\kappa} f(z) = \frac{1}{\Gamma(\kappa)} \int_0^z \frac{f(t)}{(z-t)^{1-\chi}} dt, \quad 0 \le \kappa < 1, \tag{5}$$

$$\mathfrak{D}_z^{m+\kappa} f(z) = \frac{d^m}{dz^m} \mathfrak{D}_z^{\kappa} f(z).$$

In addition to these formulas, these three operators have another well-known formula, the gamma function formula, which has been mentioned in many previous studies, as in [16], and it is:

$$\mathfrak{D}_z^{\kappa} z^{k} = \frac{\Gamma(k+1)}{\Gamma(k-\kappa+1)} z^{k-\kappa},$$

$$\mathfrak{D}_z^{-\kappa} z^{k} = \frac{\Gamma(k+1)}{\Gamma(k+\kappa+1)} z^{k+\kappa},$$

$$\mathfrak{D}_z^{s+\kappa} z^{k} = \frac{d^s}{dz^s} \mathfrak{D}_z^{\kappa} z^{k} = \frac{\Gamma(k+1)}{\Gamma(k-s-\kappa+1)} z^{k-(s+\kappa)}.$$

Since we will use the $\mathfrak{G}$ an $\mathfrak{H}$ operators in the applications of the $\mathfrak{D}_z^{\kappa}$ and $\mathfrak{D}_z^{-\kappa}$ fractional operators in our study and obtain the distinctive results, we need to mention the definition of each of them, which was mentioned in the reference [14] as follows:

$$\mathfrak{G}(z) = \left\{ \sum_{k=1}^{\infty} \frac{(1+\ell)(1+m)}{(k+\ell)(k+m)} z^{p+k-1} \right\} * f(z),$$

$$\mathfrak{H}(z) = \left\{ z^{p} \sum_{k=0}^{\infty} \frac{(\ell)_k (m)_k \left(1+\frac{p}{n}\right)_k}{\left(1+\frac{p}{n}\right)_k (r)_k} \right\} * f(z).$$

When ℓ and m are greater than negative one, n takes positive or zero values.

We have tried to provide basic and brief information about the topic, which represents the gist or the subject, so that any reader of the article can rely on it.

2- Coefficient Specifications for Functions In The Individual Class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau, \xi)$:

So, let's begin this section by proving the property that the coefficients of functions in the individual class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau, \xi)$ must possess in order to belong to it, or in other words, the necessary and sufficient condition for any function in the basic class $\mathfrak{S}(p)$ to belong to the new individual class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau, \xi)$. The details of this are given in the next theorem.

Theorem 2.1. A function f defined by type (1) in the basic class $\mathfrak{S}(p)$ is an element in the individual class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau, \xi)$ if and only if the following condition is met:

$$\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}\leq\frac{\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2},\quad(6)$$

this is when $\tau\geq 0,\ \xi<1,\vartheta\in\mathbb{R}$ and $\varsigma\geq\eta\geq 0,\nu,\varrho$ are all positive real numbers, p is a natural number, and z is naturally in the open unit disc $\mathcal{U}$.

Proof. First, let's assume that inequality (6) is true. Then, considering that $|z|=1$, we can simplify it in the following way:

$$\left|1+e^{i\vartheta}\left[\left\{\frac{(1-\tau)\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^{p}}+\frac{\tau\left(\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}}-\xi\right\}-\xi\right]\right|$$
$$-\left|1-e^{i\vartheta}\left[\left\{\frac{(1-\tau)\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^{p}}+\frac{\tau\left(\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}}-\xi\right\}-\xi\right]\right|=$$

$$\left|1+e^{i\vartheta}(1-\xi)-e^{i\vartheta}\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|$$
$$-\left|1-e^{i\vartheta}(1-\xi)+e^{i\vartheta}\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|$$

$$\geq|1+e^{i\vartheta}(1-\xi)|-|e^{i\vartheta}|\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|$$

$$-|1+e^{i\vartheta}(1-\xi)|-|e^{i\vartheta}|\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|$$

$$\geq|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|-2\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}\geq 0.$$

Which immediately gives us that

$$\mathcal{R}e\left[e^{i\vartheta}\left\{\frac{(1-\tau)\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^{p}}+\frac{\tau\left(\mathfrak{A}_{p}^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}}-\xi\right\}\right]>0,\quad(7)$$

and it decisively ends the proof of the first part

We are left with the proof of the converse part. This time, we have the hypothesis that f is an element of the class $\mathfrak{N}_{p}^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$, directly, we get

$$\left|1+e^{i\vartheta}\left(1-\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}-\xi\right)\right|$$
$$\geq\left|1-e^{i\vartheta}\left(1-\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\,\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}-\xi\right)\right|.\quad(8)$$

Or

$$\left|1+e^{i\vartheta}(1-\xi)-e^{i\vartheta}\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|$$
$$\geq\left|1-e^{i\vartheta}(1-\xi)-e^{i\vartheta}\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|. \quad (9)$$

Here, we choose the real values for the length of z and make them approach one. Let's not forget that we are within the open unit disc $\mathcal{U}$, so we can deduce the following:

$$\left|1+e^{i\vartheta}(1-\xi)\right|^{2}+\left|e^{i\vartheta}\right|^{2}\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|^{2}$$
$$-2\left|1+e^{i\vartheta}(1-\xi)\right|\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|$$

$$\geq\left|1-e^{i\vartheta}(1-\xi)\right|^{2}+\left|e^{i\vartheta}\right|^{2}\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|^{2}$$
$$+2\left|1-e^{i\vartheta}(1-\xi)\right|\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|.$$

In other words,

$$2\left|\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}z^{k}\right|\left\{\left|1-e^{i\vartheta}(1-\xi)\right|\right.$$
$$\left.+\left|1+e^{i\vartheta}(1-\xi)\right|\right\}\leq\left|1+e^{i\vartheta}(1-\xi)\right|^{2}-\left|1-e^{i\vartheta}(1-\xi)\right|^{2}$$
$$=\left\{\left|1+e^{i\vartheta}(1-\xi)\right|-\left|1-e^{i\vartheta}(1-\xi)\right|\right\}\left\{\left|1+e^{i\vartheta}(1-\xi)\right|+\left|1-e^{i\vartheta}(1-\xi)\right|\right\}.$$

After simplifying it, we get what we want, namely:

$$\sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}\leq\frac{\left\{\left|1+e^{i\vartheta}(1-\xi)\right|-\left|1-e^{i\vartheta}(1-\xi)\right|\right\}}{2}. \quad (10)$$

So, once we complete the proof of the second side of the theorem, we have fully and completely completed the proof.

It is crystal clear that we deduce the following result, which is, of course, a consequence of the previous theorem.

Corollary 2.2. Any element or function f of the different class $\mathfrak{N}_{p}^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$whose original form is (1) satisfies the following inequality:

$$a_{p+k}\leq\left\{\frac{p+\varrho}{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}\frac{p\left\{\left|1+e^{i\vartheta}(1-\xi)\right|-\left|1-e^{i\vartheta}(1-\xi)\right|\right\}}{2(p+k\tau)}, \quad (11)$$

this is when $\tau \geq 0,\ \xi < 1, \vartheta \in \mathbb{R}$ and $\varsigma \geq \eta \geq 0, \nu, \varrho$ are all positive real numbers, p is a natural number, and z is naturally in the open unit disc $\mathcal{U}$.

3- The Most Important Distortion Inequalities Associated With The Multiplier Transformation Operator:

Theorem 3.1. Every element or function f of the different class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau, \xi)$ whose original formula is (1) satisfies the following two inequalities:

$$\left| \frac{(1-\tau)\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^p} + \frac{\tau\left(\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}} \right| \geq 1 - \frac{\{|1+e^{i\vartheta}(1-\xi)| - |1-e^{i\vartheta}(1-\xi)|\}}{2}, \tag{12}$$

and

$$\left| \frac{(1-\tau)\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^p} + \frac{\tau\left(\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}} \right| \leq 1 + \frac{\{|1+e^{i\vartheta}(1-\xi)| - |1-e^{i\vartheta}(1-\xi)|\}}{2}. \tag{13}$$

This is when $\tau \geq 0,\ \xi < 1, \vartheta \in \mathbb{R}$ and $\varsigma \geq \eta \geq 0, \nu, \varrho$ are all positive real numbers, p is a natural number, and z is naturally in the open unit disc $\mathcal{U}$.

Proof. Here, we prove the first inequality, i.e., the greater than or equals case. As for the second inequality, i.e., the less than or equals case, it is similar and has the same method as the proof of the first case. Therefore,we will suffice with the proof of the first inequality, starting with the assumption

$$\begin{aligned} &\left| \frac{(1-\tau)\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^p} + \frac{\tau\left(\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}} \right| \\ &\qquad \geq 1 - \left| \sum_{k=1}^{\infty} \left\{ \frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho} \right\}^{\nu} \frac{(p+k\tau)}{p} a_{p+k} z^k \right|, \\ &\qquad \geq 1 - \sum_{k=1}^{\infty} \left\{ \frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho} \right\}^{\nu} \frac{(p+k\tau)}{p} a_{p+k} |Z|^k, \\ &\qquad \geq 1 - |z| \sum_{k=1}^{\infty} \left\{ \frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho} \right\}^{\nu} \frac{(p+k\tau)}{p} a_{p+k}. \end{aligned} \tag{14}$$

Here, we apply the Theorem 2.1 to give me

$$\left| \frac{(1-\tau)\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)}{z^p} + \frac{\tau\left(\mathfrak{A}_p^{\nu}(\varsigma,\eta,\varrho)f(z)\right)'}{pz^{p-1}} \right| \geq 1 - \frac{\{|1+e^{i\vartheta}(1-\xi)| - |1-e^{i\vartheta}(1-\xi)|\}}{2}. \tag{15}$$

Thus, I have completed its proof accurately.

Now, let us study the properties of the fractional calculus operator $\mathfrak{D}_z^{\kappa}$ by applying it and finding its distortion inequalities. We begin with the following theorem:

Theorem 3.2. Any element of the individual class $\mathfrak{N}_{p}^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$, let it be f, satisfies the following inequality:

$$|\mathfrak{D}_z^{\kappa}\mathfrak{G}(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1-\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1-\kappa+p)(p+\tau)\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}}\right]|z|\right\}, \tag{16}$$

$$|\mathfrak{D}_z^{\kappa}\mathfrak{G}(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1+\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1-\kappa+p)(p+\tau)\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}}\right]|z|\right\}, \tag{17}$$

provided $0 \leq \kappa < 0, \tau \geq 0,\ \xi < 1, \vartheta \in \mathbb{R}$ and $\varsigma \geq \eta \geq 0, \nu, \varrho$ are all positive real numbers, p is a natural number, ℓ, m are greater than -1, and z is naturally in the open unit disc $\mathcal{U}$.

Proof. We begin the proof by assuming

$$\frac{\Gamma(1+p-\kappa)}{\Gamma(1+p)}z^{\kappa-p}\mathfrak{D}_z^{\kappa}\mathfrak{G}(z) = 1 - \sum_{k=1}^{\infty}\left\{\frac{\Gamma(1+p-\kappa)\Gamma(1+p+k)(1+\ell)(1+m)}{\Gamma(1+p)\Gamma(1+p-\kappa+k)(1+\ell+k)(1+m+k)}\right\}a_{p+k}z^{k}. \tag{18}$$

And of course, the following relation is true:

$$\frac{(1+\ell)(1+m)(1+p)}{(2+\ell)(2+m)(1-\kappa+p)} > 0. \tag{19}$$

This is because the coefficient in equation (18) is a decreasing function when $\left(\frac{(1+p+k)}{(2+m+k)}\right) \geq \kappa$.

Once again, we use Theorem 2.1 to conclude that

$$\left\{\frac{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+\tau)}{p}\sum_{k=1}^{\infty}a_{p+k},$$

$$\leq \sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^{\nu}\frac{(p+k\tau)}{p}a_{p+k}, \tag{20}$$

$$\leq \frac{\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2}.$$

Thus

$$\sum_{k=1}^{\infty} a_{p+k} \leq \left\{\frac{p+\varrho}{(\varsigma\,\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu} \frac{p\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(p+\tau)}. \tag{21}$$

From this, we arrive at the desired result:

$$|\mathfrak{D}_z^{\kappa}\mathfrak{G}(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1-\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1-\kappa+p)(p+\tau)}\right] \left\{\frac{p+\varrho}{(\varsigma\,\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}, \tag{22}$$

$$|\mathfrak{D}_z^{\kappa}\mathfrak{G}(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1+\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1-\kappa+p)(p+\tau)}\right] \left\{\frac{p+\varrho}{(\varsigma\,\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}. \tag{23}$$

So, we have reached the end of the required proof in the gradual step.

If we look closely at Theorem 3.1 and consider for a moment proving the values $m = \ell-1, p = 1$, we can easily deduce the following result:

Corollary 3.3. If we reduce the individual class $\mathfrak{N}_p^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$ to the class $\mathfrak{N}_1^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$ by applying the condition $m = \ell-1, p = 1$ to it, then every element or function f of it's elements satisfies the following two inequalities:

$$|\mathfrak{D}_z^{\kappa}\mathfrak{G}(z)| \geq \frac{1}{\Gamma(2-\kappa)}|z|^{1-\kappa}\left\{1 - \left[\frac{(1+\ell)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{(3+\ell)(2-\kappa)(1+\tau)}\right]\left\{\frac{1+\varrho}{(2\varsigma\,\eta+\varsigma-\eta)+1+\varrho}\right\}^{\nu}|z|\right\}, \tag{24}$$

$$|\mathfrak{D}_z^{\kappa}\mathfrak{G}(z)| \leq \frac{1}{\Gamma(2-\kappa)}|z|^{1-\kappa}\left\{1 + \left[\frac{(1+\ell)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{(3+\ell)(2-\kappa)(1+\tau)}\right]\left\{\frac{1+\varrho}{(2\varsigma\,\eta+\varsigma-\eta)+1+\varrho}\right\}^{\nu}|z|\right\}. \tag{25}$$

We apply the fractional calculus operator $\mathfrak{D}_z^{\kappa}$ associated with the generalized operaor $\mathfrak{H}$ by finding the distortion bounds of the functions in the defined class $\mathfrak{N}_p^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$. This is detailed in the proof steps in the following theorem

Theorem 3.4. Every element or function f of the class $\mathfrak{N}_p^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$, under the condition $\ell \leq 1,\ m \leq p-\kappa+1, r \geq p+1$ and $0 \leq \kappa < 1$., satisfies the following two inequalities:

$$|\mathfrak{D}_z^\kappa \mathfrak{H}(f)(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1 - \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}, \quad (26)$$

$$|\mathfrak{D}_z^\kappa \mathfrak{G}(f)(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1 + \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}. \quad (27)$$

Proof: Based on the definition of the generalized operator $\mathfrak{H}$, it is:

$$\mathfrak{D}_z^\kappa \mathfrak{H}(z) = \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}z^{p-\kappa} - \sum_{k=1}^{\infty}\left\{\frac{(p+k\tau)}{p}\frac{\left(\frac{p}{\tau}+1\right)_k(\ell)_k(m)_k\Gamma(1+p+k)}{(r)_k\left(\frac{p}{\tau}\right)_k(1)_k\Gamma(1+p-\kappa+k)}\right\}a_{p+k}z^{p+k-\kappa}, (28)$$

Simplifying this, we obtain:

$$\frac{\Gamma(1+p-\kappa)}{\Gamma(1+p)}z^{\kappa-p}\mathfrak{D}_z^\kappa \mathfrak{H}(z) = 1 - \sum_{k=1}^{\infty}\left\{\frac{\left(\frac{p}{\tau}+1\right)_k(\ell)_k(m)_k\Gamma(1+p+k)\Gamma(1+p-\kappa)}{(r)_k\left(\frac{p}{\tau}\right)_k(1)_k\Gamma(1+p-\kappa+k)\Gamma(1+p)}\right\}a_{p+k}z^{k}. (29)$$

Since the coefficients in the previous equation are decreasing functions of k under the condition $\ell \leq 1$, $m \leq p-\kappa+1$, $r \geq p+1$ and $0 \leq \kappa < 1$, the maximum value at unity is

$$\frac{\ell m(p+1)}{r(p-\kappa+1)}.$$

Referring to inequality 6, we have:

$$\sum_{k=1}^{\infty}a_{p+k} \leq \left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}\frac{p\{|1+e^{i\vartheta}(1-\xi)|-|1+e^{i\vartheta}(1-\xi)|\}}{2(p+\tau)}, \quad (30)$$

which directly generates the required inequalities:

$$|\mathfrak{D}_z^\kappa \mathfrak{H}(f)(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1 - \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}, \quad (31)$$

$$|\mathfrak{D}_z^\kappa \mathfrak{H}(f)(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1+\left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^\nu |z|\right\}. \quad (32)$$

We have completed its proof efficiently.

We continue our application of the fractional operator $\mathfrak{D}_z^\kappa$, this time we linking it to the multiplier transformations operator $\mathfrak{A}_p^\nu$, for functions in the different class $\mathfrak{N}_p^\nu(\varsigma,\eta,\varrho,\tau,\ \xi)$ by finding the distortion bounds in the following theorem. Under the conditions that must be met.

Theorem 3.5. Every element or function f of the class $\mathfrak{N}_p^\nu(\varsigma,\eta,\varrho,\tau,\ \xi)$, under the condition $\ell\ \leq 1,\ m \leq\ p-\kappa+1, r \geq\ p+1$ and $0 \leq\ \kappa < 1$. satisfies the following two inequalities:

$$\left|\mathfrak{D}_z^\kappa\left(\mathfrak{A}_p^\nu \mathfrak{H}(f)(z)\right)\right| \geq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1-\left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]|z|\right\}, \quad (33)$$

$$\left|\mathfrak{D}_z^\kappa\left(\mathfrak{A}_p^\nu \mathfrak{H}(f)(z)\right)\right| \leq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1+\left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]|z|\right\}. \quad (34)$$

Proof: Simply take

$$\mathfrak{D}_z^\kappa\left(\mathfrak{A}_p^\nu \mathfrak{H}(f)(z)\right) = \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}z^{p-\kappa} - \sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^\nu\left\{\frac{(p+k\tau)}{p}\frac{\left(\frac{p}{\tau}+1\right)_k(\ell)_k(m)_k\Gamma(1+p+k)}{(r)_k\left(\frac{p}{\tau}\right)_k(1)_k\Gamma(1+p-\kappa+k)}\right\}a_{p+k}z^{p+k-\kappa}. \quad (35)$$

And simplify to give us:

$$\frac{\Gamma(1+p-\kappa)}{\Gamma(1+p)}z^{\kappa-p}\mathfrak{D}_z^\kappa\left(\mathfrak{A}_p^\nu \mathfrak{H}(f)(z)\right) = 1 - \sum_{k=1}^{\infty}\left\{\frac{k(\varsigma\eta(p+k)+\varsigma-\eta)+p+\varrho}{p+\varrho}\right\}^\nu\left\{\frac{(p+k\tau)}{p}\frac{\left(\frac{p}{\tau}+1\right)_k(\ell)_k(m)_k\Gamma(1+p+k)\Gamma(1+p-\kappa)}{(r)_k\left(\frac{p}{\tau}\right)_k(1)_k\Gamma(1+p-\kappa+k)\Gamma(1+p)}\right\}a_{p+k}z^k. \quad (36)$$

Since the coefficients in the previous equation are decreasing function for $\mathscr{k}$ within the limits between zero and one, under the condition $\ell \leq 1,\ m \leq p - \kappa + 1, r \geq p + 1$ and $0 \leq \kappa < 1$. then its maximum value is

$$\frac{\ell m(1+p)}{r(1-\kappa+p)} > 0. \tag{37}$$

Using inequality 6 and our previous inequality, directly gives us

$$\left|\mathfrak{D}_z^{\kappa}\left(\mathfrak{A}_p^{\nu}\mathfrak{H}(f)(z)\right)\right|$$

$$\geq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1-\left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]|z|\right\}, \tag{38}$$

$$\left|\mathfrak{D}_z^{\kappa}\left(\mathfrak{A}_p^{\nu}\mathfrak{H}(f)(z)\right)\right|$$

$$\leq \frac{\Gamma(1+p)}{\Gamma(1+p-\kappa)}|z|^{p-\kappa}\left\{1+\left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2r(1-\kappa+p)}\right]|z|\right\}. \tag{39}$$

Which the desired result.

So, we completed the proof smoothly.

Now, it is time to explore the application of the fractional calculus operator $\mathfrak{D}_z^{-\kappa}$ for the parameters $\kappa > 0$, and the use of the operators $\mathfrak{G}$ and $\mathfrak{H}$. All the details are in the following results:

Theorem 3.6. For the function f in the individual class $\mathfrak{N}_p^{\nu}(\varsigma,\eta,\varrho,\tau,\ \xi)$, then it satisfies the following two inequalities:

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)|$$

$$\geq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{\begin{array}{c}1-\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1+\kappa+p)(p+\tau)}\right]\\ \left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\end{array}\right\}, \tag{40}$$

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)|$$

$$\leq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{\begin{array}{c}1+\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1+\kappa+p)(p+\tau)}\right]\\ \left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\end{array}\right\}. \tag{41}$$

Of course, under the conditions that $\kappa > 0, \tau \geq 0,\ \xi < 1, \vartheta \in \mathbb{R}$ and $\varsigma \geq \eta \geq 0, \nu, \varrho$ are all positive real numbers, p is a natural number, ℓ, m are greater than -1, and let's not forget that z is always in the open unitary disc $\mathcal{U}$.

Proof: It is clear that it is possible to write

$$\frac{\Gamma(1+p+\kappa)}{\Gamma(1+p)} z^{-\kappa-p}\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)$$
$$= 1-\sum_{k=1}^{\infty}\left\{\frac{\Gamma(1+p+\kappa)\Gamma(1+p+k)(1+\ell)(1+m)}{\Gamma(1+p)\Gamma(1+p+\kappa+k)(1+\ell+k)(1+m+k)}\right\} a_{p+k}z^{k}. \quad (42)$$

Since the value of the coefficients in the previous series is decreasing for k, and the values are of course between zero and one, the maximum value it reaches is

$$\frac{(1+\ell)(1+m)(1+p)}{(2+\ell)(2+m)(1+\kappa+p)} > 0. \quad (43)$$

So, using Theorem 2.1 and equation 28, by substituting the maximum value, we get:

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)|$$
$$\geq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{1-\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1+\kappa+p)(p+\tau)} \left\{\frac{p+\varrho}{(\varsigma\,\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right]\right\}, \quad (44)$$

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)|$$
$$\leq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{1+\left[\frac{p(1+\ell)(1+m)(1+p)\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{2(2+\ell)(2+m)(1+\kappa+p)(p+\tau)} \left\{\frac{p+\varrho}{(\varsigma\,\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right]\right\}. \quad (45)$$

Which is the required inequalities.

Thus, we have completed the proof of the theorem.

By setting the specific values that lie at $p=1, \ell = m+1 = \kappa+2, v=0$ in the previous theorem, it immediately gives us the following result:

Corollary 3.7. The function f, which belongs to the individual class, with specific parameters $\mathfrak{N}_1^0(\varsigma,\eta,\varrho,\tau,\ \xi)$, satisfies the following tow inequalities when $\ell = m+1 = \kappa+2, v=0, p=1$:

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)| \geq \frac{1}{\Gamma(2+\kappa)}|z|^{p+\kappa}\left\{1-\left[\frac{\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{(\kappa+4)(1+\tau)}\right]|z|\right\}, \quad (46)$$

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)| \leq \frac{1}{\Gamma(2+\kappa)}|z|^{p+\kappa}\left\{1+\left[\frac{\{|1+e^{i\vartheta}(1-\xi)|-|1-e^{i\vartheta}(1-\xi)|\}}{(\kappa+4)(1+\tau)}\right]|z|\right\}. \quad (47)$$

We will discuss the application of the fractional operator $\mathfrak{D}_z^{-\kappa}$ to the general operator $\mathfrak{H}$ in the following theorem. I have decided not to write the details of its proof, since it is very similar, or in other words, the same steps of the proof of the Theorem 3.6.

Theorem 3.8. If we fix the parameters $\ell \leq 1,\ m \leq p + \kappa + 1, r \geq p + 1$ and $\kappa > 0$, then the function belonging to the individual class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau,\ \xi)$ satisfies the following two inequalities:

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{H}(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{1 - \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)| - |1-e^{i\vartheta}(1-\xi)|\}}{2r(1+\kappa+p)}\right]\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}, \tag{48}$$

$$|\mathfrak{D}_z^{-\kappa}\mathfrak{G}(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p\mp\kappa}\left\{1 + \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)| - |1\,e^{i\vartheta}(1-\xi)|\}}{2r(1+\kappa+p)}\right]\left\{\frac{p+\varrho}{(\varsigma\eta(p+1)+\varsigma-\eta)+p+\varrho}\right\}^{\nu}|z|\right\}. \tag{49}$$

We will also discuss the application of the fractional calculus operator $\mathfrak{D}_z^{-\kappa}$ to the multiplier transformations operator $\mathfrak{A}_p^{\nu}(\varsigma, \eta, \varrho)$ and the general operator $\mathfrak{H}$ under the condition of in the next theorem. For the same reason as the previous theorem, I will simply mention the theorem without the proof, since the steps for its proof are the same as those for Theorem 3.9.

Theorem 3.9. The function f belonging to the class $\mathfrak{N}_p^{\nu}(\varsigma, \eta, \varrho, \tau,\ \xi)$ satisfies the following two inequalities under the condition $\ell \leq 1,\ m \leq p + \kappa + 1, r \geq p + 1$ and $\kappa > 0$:

$$\left|\mathfrak{D}_z^{-\kappa}\left(\mathfrak{A}_p^{\nu}\mathfrak{H}(f)(z)\right)\right| \geq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{1 - \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)| - |1-e^{i\vartheta}(1-\xi)|\}}{2r(1+\kappa+p)}\right]|z|\right\}, \tag{50}$$

$$\left|\mathfrak{D}_z^{\kappa}\left(\mathfrak{A}_p^{\nu}\mathfrak{H}(f)(z)\right)\right| \leq \frac{\Gamma(1+p)}{\Gamma(1+p+\kappa)}|z|^{p+\kappa}\left\{1 + \left[\frac{\ell m(1+p)\{|1+e^{i\vartheta}(1-\xi)| - |1-e^{i\vartheta}(1-\xi)|\}}{2r(1+\kappa+p)}\right]|z|\right\}. \tag{51}$$

Conclusion

Our scientific conclusion from this research article, which presented a distinctive class of multivalent functions charachterized by multiplier transformation operator within the unit disk, is to demonstrate and study the most important fundamental properties of fractional calculus which involve multiplier transformations and other generalized operators within its applications. We obtain significant and important results in demonstrating and studying key distortion or deformation inequalities related to multiple transformation operators and fractional calculus. The most prominent of these results is the estimation of coefficients, and other exicting results. These results open avenues for future research, either through the study of other interesting results that compement this research topic or through the study of other properties that contribute effectively to development and enrichment of this field of complex analysis.

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