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On N-(n, m)-D-quasi operator on Hilbert space

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Abstract:

In this paper, We provide a novel class of bounded linear operators on a Hilbert space that we refer to as N-(n, m)-D-quasi operator, defined through an operator identity that relies on the Drazin inverse. We study their fundamental structure and prove their stability undertaking powers, restriction to invariant subspaces, and unitary equivalence. We also analyze their behavior under algebraic operations such as sums and products and show that the class is preserved under direct sums and tensor products, supported by examples highlighting the main features of this class.

Keywords: N-(n,m) $-$D-quasi operator, N quasi D-operator, Drazin inverse.

1-Introduction Section

Let $B(H)$ be the algebra of all bounded linear operators on H, assuming that H is a Hilbert space. In recent years, there has been increasing interest in extending and generalizing classes of operators on Hilbert spaces. This has been achieved by introducing new operator classes such as N quasi-D-operator (see [1]) and by relaxing certain classical conditions associated with these classes. An operator $T_{op} \in B(H)$ is reported to be normal $T_{op}T_{op}^{*} = T_{op}^{*}T_{op}$, self-adjoint if $T_{op} = T_{op}^{*}$[2]. An operator T_{op} is reported to class (Q) if $T_{op}^{*2}T_{op}^{n2} = \left(T_{op}^{*}T_{op}\right)^{2}$. [3] , M Quasi class (Q) if $T_{op}\left(T_{op}^{*2}T_{op}^{2}\right) = \mathrm{M}\left(T_{op}^{*}T_{op}\right)^{2}T_{op}$. [4] , N quasi $\mathrm{D}-\mathrm{Operator}$ if if $T_{op}\left(T_{op}^{*2}\left(T_{op}^{D}\right)^{2}\right) = \mathrm{M}\left(T_{op}^{*}T_{op}^{D}\right)^{2}T_{op}$. [1], $(\mathrm{K}-\mathrm{N})$ $\mathrm{Quasi}-\mathrm{normal\ Operator}$ if $Ţ^{k}\left(T_{op}^{*}T_{op}\right) = N\left(T_{op}^{*}T_{op}\right)T_{op}^{k}$. [5], $(\mathrm{K}-\mathrm{N})$ $\mathrm{Quasi\ n\ normal\ Operator}$ if $T_{op}^{k}\left(T_{op}^{*}T_{op}^{n}\right) = N\left(T_{op}^{*}T_{op}^{n}\right)T_{op}^{k}$. [6]. And others [12 $-$ 17].

Four sections make up the paper. The work is introduced in Section 1, the fundamental concepts are covered in Section 2, the majority of the features of the N-(n,m) $-$D-quasi operator are studied in Section 3, and a conclusion is finally drawn.

2- Preliminaries

Definition 2.1.[2] Let Ẋ be a vector space over the field $\mathcal{F}$. An inner product is a mapping from Ẋ×Ẋ onto the field that has certain requirements. For all á ∈ Ẋ, we must have ⟨á , á⟩ ≥ 0In case ⟨á , á⟩ = 0 if and only if á = 0For all á, $\acute{b}$ ∈ Ẋ, holds ⟨á , $\acute{b}$⟩ = $\overline{\langle \acute{b} , á\rangle}$

1. á, $\acute{b}$, $\acute{c}$ ∈ Ẋ, holds $\langle \alpha á + \beta\acute{b}, \acute{c}\rangle = \alpha\langle á, \acute{c}\rangle + \beta\langle \acute{b}, \acute{c}\rangle$, with $\alpha, \beta \in \mathcal{F}$

The pair (Ẋ, ⟨ , ⟩) be a pre-Hilbert space.

Definition 2.2. [7, 8, 9] Suppose that $A_{op} \in B(H)$. The Drazin inverse of A_{op} is the operator $A_{op}^D: H \to H$ such that this condition holds:

1. $A_{op}A_{op}^D = A_{op}^D A_{op}$.
2. $A_{op}^D A_{op} A_{op}^D = A_{op}^D$.
3. $A_{op}^{s+1}A_{op}^D = A_{op}^s$. for at least one positive integer s.

The least nonnegative integer r is referred to as the index of A_{op}, indicated by $ind(A_{op})$ and $ind(A_{op}) = 0$ if and only if $A_{op}^D = A_{op}^{-1}$.

Definition 2.3.[10]Assume that $A_{op} \in B(H)$ be Drazin operator then , A_{op} is called an (n,m) -D-quasi operator if
$A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n}) = (A_{op}^*(A_{op}^D)^n)^2 A_{op}^m$, where the integer n, m ≥ 1.

The following lemma states several fundamental properties of Drazin operator established in [7,8].

Lemma 2.4. Let $A_{op}, T_{op} \in B(H)$ be two Drazin invertible operators:

1. $\left(A_{op}^*\right)^D = \left(A_{op}^D\right)^*$.
2. $(A_{op}^D)^\lambda = (A_{op}^\lambda)^D$, $\lambda = 1,2,3, \ldots$.
3. If $A_{op}T_{op} = T_{op}A_{op}$ then $\left(A_{op}T_{op}\right)^D = T_{op}^D A_{op}^D = A_{op}^D T_{op}^D, A_{op}T_{op}^D = T_{op}^D A_{op}$.
4. $(T_{op}^{-1}A_{op}T_{op})^D = T_{op}^{-1}A_{op}^D T_{op}$.
5. If $A_{op}T_{op} = T_{op}A_{op} = 0_{op}$ then $A_{op}^D + T_{op}^D = (A_{op} + T_{op})^D$

3- Main Results

Definition 3.1. let $A_{op} \in B(H)$ be Drazin operator then A_{op} is called an N-(n, m)-D-quasi operator if

$$A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n}) = N\,(A_{op}^*(A_{op}^D)^n)^2 A_{op}^m \tag{1}$$

where N is a bounded operator on H and m, n ≥ 1.
The class of all N-(n, m)-D-quasi operator denoted by $N - [\mathrm{D}_{\mathrm{n,m}}]$.

The following proposition presents the most important generalizations of these operators within this type of study.

Proposition 3.2. Suppose that $A_{op} \in N - [\mathrm{D}_{\mathrm{n,m}}]$ then
$\left(A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n})\right)^r = \left(N\,(A_{op}^*(A_{op}^D)^n)^2 A_{op}^m\right)^r$, $r \in \mathbb{N}$.

Proof.
By using the principle of mathematical induction,
When r = 1 then

$$A_{op}^{m}(A_{op}^{*2}(A_{op}^{D})^{2n}) = N\,(A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}$$

Suppose that the statement holds at k, so became

$$\left(A_{op}^{m}(A_{op}^{*2}(A_{op}^{D})^{2n})\right)^{k} = \left(N\,(A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}\right)^{k}$$

Now,

$$\begin{aligned}\left(A_{op}^{m}(A_{op}^{*2}(A_{op}^{D})^{2n})\right)^{k+1} &= A_{op}^{m}(A_{op}^{*2}(A_{op}^{D})^{2n})\left(A_{op}^{m}(A_{op}^{*2}(A_{op}^{D})^{2n})\right)^{k}\\ &= \left(N\,(A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}\right)\left(N\,(A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}\right)^{k}\\ &= \left(N\,(A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}\right)^{k+1}\end{aligned}$$

Hence,

$$\left(A_{op}^{m}(A_{op}^{*2}(A_{op}^{D})^{2n})\right)^{r} = \left(N\,(A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}\right)^{r},\ r\in\mathbb{N}. \tag{2}$$

Proposition 3.3. Suppose that $T_{op}\in N-[\mathrm{D}_{n,m}]$ then $T_{op_{|M}}\in N-[\mathrm{D}_{n,m}]$ such that $\mathrm{N}_{|M}=N$, M is closed subspace of H.

Proof.

$$\begin{aligned}(T_{op_{|M}})^{m}\left((T_{op_{|M}})^{*2}\left(\left(T_{op_{|M}}\right)^{D}\right)^{2n}\right) &= (T_{op_{|M}})^{m}\left((T_{op_{|M}})^{*}(T_{op_{|M}})^{*}\left(T_{op_{|M}}^{D}\right)^{n}\left(T_{op_{|M}}^{D}\right)^{n}\right)\\ &= (T_{op_{|M}})^{m}(T_{op_{|M}}^{*})(T_{op_{|M}}^{*})\left({(T_{op}^{D})^{n}}_{|M}\right)\left({(T_{op}^{D})^{n}}_{|M}\right)\\ &= (T_{op_{|M}})^{m}(T_{op_{|M}}^{*})^{2}\left({(T_{op}^{D})^{n}}_{|M}\right)^{2}\\ &= (T_{op_{|M}})^{m}(T_{op_{|M}}^{*2})\left({(T_{op}^{D})^{2n}}_{|M}\right)\\ &= \left(T_{op}^{m}\left(T_{op}^{*2}(T_{op}^{D})^{2n}\right)\right)_{|M} \qquad ,T_{op}\in \mathrm{N}-[\mathrm{D}_{n,m}]\\ &= \left((\mathrm{N}T_{op}^{*}(T_{op}^{D})^{n})^{2}T_{op}^{m}\right)_{|M}\\ &= \mathrm{N}\left((T_{op_{|M}})^{*}((T_{op_{|M}})^{D})^{n}\right)\left((T_{op_{|M}})^{*}((T_{op_{|M}})^{D})^{n}\right)(T_{op_{|M}})^{m}\\ &= \mathrm{N}\left((T_{op_{|M}})^{*}((T_{op_{|M}})^{D})^{n}\right)^{2}(T_{op_{|M}})^{m}\end{aligned} \tag{3}$$

Hence, $T_{op_{|M}}\in \mathrm{N}-[\mathrm{D}_{n,m}]$. □

Proposition 3.4. Suppose that $\mathrm{A_{op}}\in \mathrm{B(H)}$ and for any $\mathrm{T_{op}}=\mathrm{S_{op}A_{op}S_{op}^{*}}\in\mathrm{B(H)}$ which is unitary equivalent to $\mathrm{A_{op}}$ then, $\mathrm{A_{op}}\in\mathrm{N}-[\mathrm{D}_{n,m}]$ if and only if $\mathrm{T_{op}}\in\mathrm{N}-[\mathrm{D}_{n,m}]$ such that $\mathrm{T_{op}S_{op}}=\mathrm{S_{op}T_{op}}$.

Proof.

Let $\mathrm{A_{op}}\in\mathrm{N}-[\mathrm{D}_{n,m}]$ and $\mathrm{T_{op}}$ is unitary equivalent to $\mathrm{A_{op}}$ then, $\mathrm{T_{op}}=\mathrm{S_{op}A_{op}S_{op}^{*}}$, such that $\mathrm{S_{op}}$ is unitary operator. Therefore,

$$\begin{aligned}\mathrm{T_{op}^{m}}(\mathrm{T_{op}^{*2}}(\mathrm{T_{op}^{D}})^{2n}) &= (\mathrm{S_{op}A_{op}S_{op}^{*}})^{m}((\mathrm{S_{op}A_{op}S_{op}^{*}})^{*2}((\mathrm{S_{op}A_{op}S_{op}^{*}})^{D})^{2n})\\ &= (\mathrm{S_{op}A_{op}^{m}S_{op}^{*}})(\mathrm{S_{op}A_{op}^{*2}S_{op}^{*}})(\mathrm{S_{op}}(\mathrm{A_{op}^{D}})^{2n}\mathrm{S_{op}^{*}}) \qquad (\mathrm{S_{op}S_{op}^{*}}=\mathrm{I_{op}})\\ &= \mathrm{S_{op}}(\mathrm{A}^{*}(\mathrm{A_{op}^{D}})^{n})^{2}\mathrm{A_{op}^{m}S_{op}^{*}} \qquad \mathrm{A_{op}}\in\mathrm{N}-[\mathrm{D}_{n,m}]\\ &= \mathrm{N\,S_{op}A_{op}^{*}}(\mathrm{A_{op}^{D}})^{n}\mathrm{A_{op}^{*}}(\mathrm{A_{op}^{D}})^{n}\mathrm{A_{op}^{m}S_{op}^{*}}\\ &= \mathrm{N}(\mathrm{S_{op}A_{op}S_{op}^{*}})^{*}(\mathrm{S_{op}A_{op}^{D}S_{op}^{*}})^{n}(\mathrm{S_{op}A_{op}S_{op}^{*}})^{*}(\mathrm{S_{op}A_{op}^{D}S_{op}^{*}})^{n}(\mathrm{S_{op}A_{op}S_{op}^{*}})^{m}\\ &= \mathrm{N\,T_{op}^{*}}(\mathrm{T_{op}^{D}})^{n}\mathrm{T_{op}^{*}}(\mathrm{T_{op}^{D}})^{n}\mathrm{T_{op}^{m}}\\ &= \mathrm{N}(\mathrm{T_{op}^{*}}(\mathrm{T_{op}^{D}})^{n})^{2}\mathrm{T_{op}^{m}}\end{aligned} \tag{4}$$

Thus, $\mathrm{T_{op}}\in\mathrm{N}-[\mathrm{D}_{n,m}]$.

Suppose $\mathrm{T_{op}}\in\mathrm{N}-[\mathrm{D}_{n,m}]$ then

$$\mathrm{T_{op}^{m}}(\mathrm{T_{op}^{*2}}(\mathrm{T_{op}^{D}})^{2n}) = \mathrm{N}(\mathrm{T_{op}^{*}}(\mathrm{T_{op}^{D}})^{n})^{2}\mathrm{T_{op}^{m}}$$

$\Rightarrow (S_{op}A_{op}S_{op}^*)^m((S_{op}A_{op}S_{op}^*)^{*2}((S_{op}A_{op}S_{op}^*)^D)^{2n})$
$= N(S_{op}A_{op}S_{op}^*)^*(S_{op}A_{op}^D S_{op}^*)^n(S_{op}A_{op}S_{op}^*)^*(S_{op}A_{op}^D S_{op}^*)^n(S_{op}A_{op}S_{op}^*)^m$
$\Rightarrow S_{op}(A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n})S_{op}^* = N\, S_{op}(A_{op}^*(A_{op}^D)^n)^2 A_{op}^m S_{op}^*$
$\Rightarrow A_{op}^m(A_{op}^{*2}(A_{op}^D)^{2n}) = N(A_{op}^*(A_{op}^D)^n)^2 A_{op}^m.$ (5)
Therefore, $A_{op} \in N-[D_{n,m}]$. □

Proposition 3.5. Let $T_{op} \in N-[D_{n,m}]$ and T_{op}^D is normal then $T_{op}^D \in N-[D_{n,m}]$, when the following are holds

1. N is equal to N^D
2. N commute with $(T_{op}^*(T_{op}^D)^n)^2 T_{op}^m$.

Proof.
Since $T_{op}^D$is normal then $T_{op}^D(T_{op}^D)^* = (T_{op}^D)^* T_{op}^D$ and from Definition2.2, $T_{op}T_{op}^D = T_{op}^D T_{op}$ by Fuglede Theorem [11] then $T_{op}(T_{op}^D)^* = (T_{op}^D)^* T_{op}$.
Now, since $T_{op} \in N-[D_{n,m}]$ then

$$\Rightarrow T_{op}^m(T_{op}^{*2}(T_{op}^D)^{2n}) = N(T_{op}^*(T_{op}^D)^n)^2 T_{op}^m$$
$$\Rightarrow T_{op}^m(T_{op}^{*2}(T_{op}^D)^{2n}) = (T_{op}^*(T_{op}^D)^n)^2 T_{op}^m\, N$$

By taking the Drazin of both sides we have

$$\Rightarrow T_{op}^{2n}(T_{op}^D)^{*2}(T_{op}^D)^m = N^D (T_{op}^D)^m (T_{op}^n(T_{op}^D)^*)^2$$
$$\Rightarrow (T_{op}^D)^m T_{op}^{2n}(T_{op}^D)^{*2} = N\left((T_{op}^D)^* T_{op}^n\right)^2 (T_{op}^D)^m$$
$$\Rightarrow (T_{op}^D)^m \left((T_{op}^D)^{*2}\left((T_{op}^D)^D\right)^{2n}\right) = N\left((T_{op}^D)^*\left((T_{op}^D)^D\right)^n\right)^2 (T_{op}^D)^m \quad (6)$$

Hence $T_{op}^D \in N-[D_{n,m}]$. □

Proposition 3.6. Let $T_{op} \in N-[D_{n,m}]$ is normal then $T_{op}^* \in N-[D_{n,m}]$ when the following are holds

1. N is equal to N^*
2. N commute with $(T_{op}^*(T_{op}^D)^n)^2 T_{op}^m$.

Proof.
Given that T_{op} is normal and from Definition 2.2, $T_{op}T_{op}^D = T_{op}^D T_{op}$ by Fuglede Theorem [11] then $T_{op}^* T_{op}^D = T_{op}^D T_{op}^*$ and $T_{op}^*(T_{op}^*)^D = (T_{op}^*)^D T_{op}^*$. Now

$$T_{op}^m(T_{op}^{*2}T_{op}^D)^{2n}) = N\,(T_{op}^*(T_{op}^D)^n)^2 T_{op}^m$$
$$\Rightarrow T_{op}^m(T_{op}^{*2}(T_{op}^D)^{2n}) = (T_{op}^*(T_{op}^D)^n)^2 T_{op}^m\, N$$

By taking the adjoint of both sides we have

$$\Rightarrow \left((T_{op}^D)^*\right)^{2n} T_{op}^2 (T_{op}^*)^m = N^*(T_{op}^*)^m \left(\left((T_{op}^D)^*\right)^n T_{op}\right)^2$$
$$\Rightarrow (T_{op}^*)^m T_{op}^2 \left((T_{op}^*)^D\right)^{2n} = N\, T_{op}\left((T_{op}^*)^D\right)^n T_{op}\left((T_{op}^*)^D\right)^n (T_{op}^*)^m$$
$$\Rightarrow (T_{op}^*)^m \left((T_{op}^*)^{*2}\left((T_{op}^*)^D\right)^{2n}\right) = N\left((T_{op}^*)^*\left((T_{op}^*)^D\right)^n\right)^2 (T_{op}^*)^m \quad (7)$$

Hence $T_{op}^* \in N-[D_{n,m}]$. □

Theorem 3.7. Let $T_{op} \in B(H)$ and for every non-zero $\lambda \in \mathcal{F}$ then, $T_{op} \in N-[D_{n,m}]$ if and only if $\lambda T_{op} \in N-[D_{n,m}]$.
Proof.
Assume that $T_{op} \in N-[D_{n,m}]$

$$\begin{aligned}(\lambda T_{op})^m\big((\lambda T_{op})^{*2}((\lambda T_{op})^D)^{2n}\big) &= \lambda^m T^m\big((\bar{\lambda})^2 T_{op}^{*2}\lambda^{2n}(T_{op}^D)^{2n}\big)\\ &= \lambda^m(\bar{\lambda})^2\lambda^{2n}T_{op}^m\big(T_{op}^{*2}(T_{op}^D)^{2n}\big), \qquad \lambda\in\mathcal{F}\\ &= \lambda^m(\bar{\lambda})^2\lambda^{2n}N(T_{op}^*(T_{op}^D)^n)^2T_{op}^m\\ &= N\big((\lambda T_{op})^*(\lambda T_{op}^D)^n\big)^2(\lambda T_{op})^m. \qquad (8)\end{aligned}$$

Suppose that $\lambda T_{op}\in[D_{n,m}]$ and $\lambda\neq 0$,
Therefore,

$$(\lambda T_{op})^m\big((\lambda T_{op})^{*2}((\lambda T_{op})^D)^{2n}\big) = N\big((\lambda T_{op})^*(\lambda T_{op}^D)^n\big)^2(\lambda T_{op})^m$$

$$\Rightarrow \lambda^m T_{op}^m\big((\bar{\lambda})^2T_{op}^{*2}\lambda^{2n}(T_{op}^D)^{2n}\big) = N\big((\lambda T_{op})^*(\lambda T_{op}^D)^n\big)\big((\lambda T_{op})^*(\lambda T_{op}^D)^n\big)\lambda^m T_{op}^m$$

$$\Rightarrow \lambda^m(\bar{\lambda})^2\lambda^{2n}T_{op}^m\big(T_{op}^{*2}(T_{op}^D)^{2n}\big) = N\big(\bar{\lambda}T_{op}^*\lambda^n\,(T_{op}^D)^n\big)\big(\bar{\lambda}T_{op}^*\lambda^n\,(T_{op}^D)^n\big)\lambda^m T_{op}^m$$

$$\Rightarrow \lambda^m(\bar{\lambda})^2\lambda^{2n}T_{op}^m\big(T_{op}^{*2}(T_{op}^D)^{2n}\big) = \lambda^m(\bar{\lambda})^2\lambda^{2n}\,N\,(T_{op}^*(T_{op}^D)^n)^2T_{op}^m$$

From some basic rules

$$\Rightarrow T_{op}^m\big(T_{op}^{*2}(T_{op}^D)^{2n}\big) = N(T_{op}^*(T_{op}^D)^n)^2T_{op}^m. \qquad (9)$$

Hence $T_{op}\in N-[D_{n,m}]$. □

Remark 3.8. In Theorem 3.7, if $\lambda=0$ then $\lambda T_{op}\in N-[D_{n,m}]$ but T_{op} not necessarily belong to $N-[D_{n,m}]$.

Theorem 3.9. Let $T_{op}, A_{op}\in N-[D_{n,m}]$ and T_{op}, A_{op} are normal operator such that $T_{op}^m A_{op}^*(A_{op}^D)^n = A_{op}^m T_{op}^*(T_{op}^D)^n = T_{op}A_{op} = A_{op}T_{op} = 0_{op}$ then $T_{op}+A_{op}\in N-[D_{n,m}]$.
Proof.
Since $T_{op}A_{op}=A_{op}T_{op}=0_{op}$ then $(T_{op}+A_{op})^m = T_{op}^m + A_{op}^m$ and by lemma 2.4(3) We've got $T_{op}^D A_{op} = A_{op}T_{op}^D = A_{op}^D T_{op} = T_{op}A_{op}^D = 0_{op}$ Now, since T_{op} is normal by Fuglede Theorem [11], $T_{op}^*A_{op}^D = A_{op}^D T_{op}^* = 0_{op}$. Similarly, $T_{op}^D A_{op}^* = A_{op}^* T_{op}^D = 0_{op}$, it follows that $T_{op}^{*2}(A_{op}^D)^{2n} = A_{op}^{*2}(T_{op}^D)^{2n} = 0_{op}$. Thus

$$\begin{aligned}&(T_{op}+A_{op})^m\big((T_{op}+A_{op})^{*2}((T_{op}+A_{op})^D)^{2n}\big)\\ &= (T_{op}+A_{op})^m(T_{op}^{*2}+A_{op}^{*2})\big((T_{op}^D)^{2n}+(A_{op}^D)^{2n}\big)\\ &= (T_{op}+A_{op})^m\big(T_{op}^{*2}(T_{op}^D)^{2n}+T_{op}^{*2}(A_{op}^D)^{2n}+A_{op}^{*2}(T_{op}^D)^{2n}+A_{op}^{*2}(A_{op}^D)^{2n}\big)\\ &= (T_{op}^m+A_{op}^m)\big(T_{op}^{*2}(T_{op}^D)^{2n}+A_{op}^{*2}(A_{op}^D)^{2n}\big)\\ &= T_{op}^mT_{op}^{*2}(T_{op}^D)^{2n}+T_{op}^mA_{op}^{*2}(A_{op}^D)^{2n}+A_{op}^mT_{op}^{*2}(T_{op}^D)^{2n}+A_{op}^mA_{op}^{*2}(A_{op}^D)^{2n}\\ &= T_{op}^mT_{op}^{*2}(T_{op}^D)^{2n}+A_{op}^mA_{op}^{*2}(A_{op}^D)^{2n}\\ &= N\,(T_{op}^*(T_{op}^D)^n)^2T_{op}^m + N\,(A_{op}^*(A_{op}^D)^n)^2A_{op}^m, \qquad T_{op}, A_{op}\in N-[D_{n,m}]\\ &= N\Big(\big((T_{op}^*(T_{op}^D)^n)^2+(A_{op}^*(A_{op}^D)^n)^2\big)(T_{op}^m+A_{op}^m)\Big)\\ &= N\Big((T_{op}^*(T_{op}^D)^n+A_{op}^*(A_{op}^D)^n)(T_{op}^*(T_{op}^D)^n+A_{op}^*(A_{op}^D)^n)(T_{op}+A_{op})^m\Big)\\ &= N\Big(\big((T_{op}+A_{op})^*((T_{op}+A_{op})^D)^n\big)\big((T_{op}^*+A_{op}^*)((T_{op}+A_{op})^D)^n\big)(T_{op}+A_{op})^m\Big)\\ &= N\Big((T_{op}+A_{op})^*\big((T_{op}+A_{op})^D\big)^n\Big)^2(T_{op}+A_{op})^m \qquad (10)\end{aligned}$$

Hence, $T_{op}+A_{op}\in N-[D_{n,m}]$. □

Theorem 3.10. Let $T_{op}, A_{op} \in N-[D_{n,m}]$ and T_{op}, A_{op} are normal operator such that $T_{op}^{m}A_{op}^{*}(A_{op}^{D})^{n} = A_{op}^{m}T_{op}^{*}(T_{op}^{D})^{n} = T_{op}A_{op} = A_{op}T_{op} = 0_{op}$ then then $T_{op} - A_{op} \in N-[D_{n,m}]$.
Proof.
The proof is a direct application of Theorem 3.9.

Theorem 3.11. Let $T_{op} \in N-[D_{n,m}]$ and $A_{op} \in I_{op}-[D_{n,m}]$ such that I_{op} is the identity operator then $T_{op}A_{op} \in N-[D_{n,m}]$. If the following are holds:

1. $T_{op}A_{op} = A_{op}T_{op}$
2. $T_{op}^{*2}A_{op}^{m} = A_{op}^{m}T^{*2}$

Proof.
Since $T_{op}A_{op} = A_{op}T_{op}$ and $T_{op}^{*2}A_{op}^{m} = A_{op}^{m}T_{op}^{*2}$ then through a lemma 2.4(3) We've got $A_{op}^{*}T_{op}^{D} = T_{op}^{D}A_{op}^{*}$ and $T_{op}^{D}A_{op} = A_{op}T_{op}^{D}$ it is following that $(T_{op}^{D})^{2n}A_{op}^{m} = A_{op}^{m}(T_{op}^{D})^{2n}$ and $(T_{op}^{D})^{2n}A_{op}^{m} = A_{op}^{m}(T_{op}^{D})^{2n}$.
Therefore,

$$
\begin{aligned}
&(T_{op}A_{op})^{m}((T_{op}A_{op})^{*2}((T_{op}A_{op})^{D})^{2n}) \\
&= (T_{op}A_{op})^{m}((A_{op}T_{op})^{*2}((A_{op}T_{op})^{D})^{2n}) \\
&= T_{op}^{m}A_{op}^{m}\left(T_{op}^{*2}A_{op}^{*2}\right)\left(T_{op}^{D}\right)^{2n}\left(A_{op}^{D}\right)^{2n} \\
&= T_{op}^{m}\left(A_{op}^{m}T_{op}^{*2}\right)\left(A_{op}^{*2}\left(T_{op}^{D}\right)^{2n}\right)\left(A_{op}^{D}\right)^{2n} \\
&= T_{op}^{m}\left(T_{op}^{*2}A_{op}^{m}\right)\left((T_{op}^{D})^{2n}A_{op}^{*2}\right)\left(A_{op}^{D}\right)^{2n} \\
&= T_{op}^{m}T_{op}^{*2}\left(A_{op}^{m}(T_{op}^{D})^{2n}\right)\left(A_{op}^{*2}\left(A_{op}^{D}\right)^{2n}\right) \\
&= T_{op}^{m}T_{op}^{*2}\left((T_{op}^{D})^{2n}A_{op}^{m}\right)\left(A_{op}^{*2}\left(A_{op}^{D}\right)^{2n}\right) \\
&= \left(T_{op}^{m}T_{op}^{*2}(T_{op}^{D})^{2n}\right)\left(A_{op}^{m}A_{op}^{*2}\left(A_{op}^{D}\right)^{2n}\right) \\
&= N\left((T_{op}^{*}(T_{op}^{D})^{n})^{2}T_{op}^{m}\right)\left((A_{op}^{*}(A_{op}^{D})^{n})^{2}A_{op}^{m}\right) \quad , \qquad T_{op} \in N-[D_{n,m}] \\
&= N\, T_{op}^{*2}A_{op}^{*2}(T_{op}^{D})^{2n}\left(A_{op}^{D}\right)^{2n}T_{op}^{m}A_{op}^{m} \\
&= N\, (T_{op}A_{op})^{*\,2}\left(\left(T_{op}A_{op}\right)^{D}\right)^{2n} T_{op}^{m}A_{op}^{m} \\
&= N\, \left((T_{op}A_{op})^{*}\left(\left(T_{op}A_{op}\right)^{D}\right)^{n}\right)^{2}(T_{op}A_{op})^{m} \qquad (11)
\end{aligned}
$$

Hence, $T_{op}A_{op} \in N-[D_{n,m}]$. □

Theorems (3.9), (3.10), and (3.11) are not always true in general, as the following example shows.

Example 3.12. Let A_{op}, T_{op}, S_{op} and G_{op} be operators on the Hilbert space $\mathbb{C}^{2}$ and N is the identity operator. Then

1. Let $A_{op} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $S_{op} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

Therefore,
$S_{op}^{*} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$, $A_{op}^{*} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $S_{op}^{D} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $A_{op}^{D} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$
We note that $A_{op}S_{op} \neq S_{op}A_{op}$ and $A_{op}, S_{op} \in N-[D_{1,2}]$.
But it is easy to compute that $A_{op} + S_{op} \notin N-[D_{1,2}]$ and $A_{op} - S_{op} \notin N-[D_{1,2}]$

2. Let $T_{op} = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$ and $G_{op} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ then

$G_{op}^{*} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $T_{op}^{*} = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$ and $G_{op}^{D} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $T_{op}^{D} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}$

Therefore, $T_{op}G_{op} \neq G_{op}T_{op}$ and $T_{op}, G_{op} \in N-[D_{1,2}]$.
But it is easy to compute that $T_{op}G_{op} \notin N-[D_{1,2}]$.

Theorem 3.13. Let $U = \{T_{op}, T_{op}: H \to H \text{ is}$ N- (n, m)-D-quasi operator on H$\} \subseteq B(H)$ then U is closed subset of $B(H)$ which is closed under scaler multiplication.
Proof.
Assume that (T_k) be a sequence of N- (n, m)-D-quasi operator such that $T_k \to T_{op}$ in U. To show that $T_{op} \in U$. Now

$$\left\|T_{op}^{m}(T_{op}^{*2}(T_{op}^{D})^{2n}) - N(T_{op}^{*}(T_{op}^{D})^{n})^{2}T_{op}^{m}\right\|$$

$$= \left\|T_{op}^{m}(T_{op}^{*2}(T_{op}^{D})^{2n}) - T_k{}^{m}(T_k{}^{*2}(T_k{}^{D})^{2n}) + T_k{}^{m}(T_k{}^{*2}(T_k{}^{D})^{2n}) - N(T_{op}^{*}(T_{op}^{D})^{n})^{2}T_{op}^{m}\right\|$$

$$\leq \left\|T_{op}^{m}(T_{op}^{*2}(T_{op}^{D})^{2n}) - T_k{}^{m}(T_k{}^{*2}(T_k{}^{D})^{2n})\right\| + \left\|T_k{}^{m}(T_k{}^{*2}(T_k{}^{D})^{2n}) - N(T_{op}^{*}(T_{op}^{D})^{n})^{2}T_{op}^{m}\right\| \to 0$$

When $k \to \infty$. Hence $T_{op} \in U$. □

Theorem 3.14. Let $A_{op_1}, A_{op_2}, \ldots . A_{op_r} \in N-[D_{n,m}]$ then

1. $\bigoplus_{i=1}^{r} A_{op_i} \in N-[D_{n,m}]$.
2. $\bigotimes_{i=1}^{r} A_{op_i} \in N-[D_{n,m}]$.

Proof.

1. $$\left(\bigoplus_{i=1}^{r} A_{op_i}\right)^{m}\left(\left(\bigoplus_{i=1}^{r} A_{op_i}\right)^{*2}\left(\left(\bigoplus_{i=1}^{r} A_{op_i}\right)^{D}\right)^{2n}\right)$$
$$= \left(\bigoplus_{i=1}^{r} A_{op_i}^{m}\right)\left(\bigoplus_{i=1}^{r} A_{op_i}^{*2}\right)\left(\bigoplus_{i=1}^{r}\left(A_{op_i}^{D}\right)^{2n}\right)$$
$$= \bigoplus_{i=1}^{r} A_{op_i}^{m} A_{op_i}^{*2}\left(A_{op_i}^{D}\right)^{2n}$$
$$= \bigoplus_{i=1}^{r} N(A_{op_i}^{*}(A_{op_i}^{D})^{n})^{2} A_{op_i}^{m}$$
$$= N\left(\bigoplus_{i=1}^{r} A_{op_i}^{*}(A_{op_i}^{D})^{n}\right)^{2}\left(\bigoplus_{i=1}^{r} A_{op_i}^{m}\right) \tag{12}$$

2. Let $x = \bigotimes_{i=1}^{r} x_i \in H \otimes \ldots \otimes H$, it is following that

$$\left(\bigotimes_{i=1}^{r} A_{op_i}\right)^{m}\left(\left(\bigotimes_{i=1}^{r} A_{op_i}\right)^{*2}\left(\left(\bigotimes_{i=1}^{r} A_{op_i}\right)^{D}\right)^{2n}\right)(x)$$
$$= \left(\bigotimes_{i=1}^{r} A_{op_i}^{m}\right)\left(\bigotimes_{i=1}^{r} A_{op_i}^{*2}\right)\left(\bigotimes_{i=1}^{r}\left(A_{op_i}^{D}\right)^{2n}\right)\left(\bigotimes_{i=1}^{r} x_i\right)$$
$$= \bigotimes_{i=1}^{r} A_{op_i}^{m} A_{op_i}^{*2}\left(A_{op_i}^{D}\right)^{2n}(x_i)$$
$$= \bigotimes_{i=1}^{r} N(A_{op_i}^{*}(A_{op_i}^{D})^{n})^{2} A_{op_i}^{m}(x_i)$$
$$= N\left(\bigotimes_{i=1}^{r} A_{op_i}^{*}(A_{op_i}^{D})^{n}(x_i)\right)^{2}\left(\bigotimes_{i=1}^{r} A_{op_i}^{m}(x_i)\right)$$
$$= N\left(\bigotimes_{i=1}^{r} A_{op_i}^{*}(A_{op_i}^{D})^{n}\right)^{2}\left(\bigotimes_{i=1}^{r} A_{op_i}\right)^{m}\left(\bigotimes_{i=1}^{r} x_i\right) \tag{13}$$

The proof is finished. □

The converse of Theorem 3.14 (2) does not generally hold, as the following example shows.

Example 3.15. Let $A_{op} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $T_{op} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ be operators on the Hilbert space $\mathbb{C}^2$ and N is the identity operator. Then

$$\mathrm{A_{op}} \otimes \mathrm{T_{op}} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \left(\mathrm{A_{op}} \otimes \mathrm{T_{op}}\right)^{*} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \text{ and}$$

$$\left(\mathrm{A_{op}} \otimes \mathrm{T_{op}}\right)^{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to compute that $\mathrm{A_{op}} \otimes \mathrm{T_{op}} \in N - [\mathrm{D_{1,2}}]$ but $\mathrm{A_{op}} \notin N - [\mathrm{D_{1,2}}]$.

4- Conclusion

The findings show that the N-(n,m)-D-quasi operator property embodies a genuine structural compatibility between an operator $\mathrm{A_{op}}$, its adjoint $\mathrm{A_{op}^{*}}$, and the Drazin inverse $\mathrm{A_{op}^{D}}$. This property is preserved under natural transformations such as unitary equivalence and restriction to invariant subspace because it behaves as an essential feature of the operator. The study also shows that the property is sensitive to how operators are assembled: it remains stable under standard constructions and algebraic combinations, including direct sums and tensor products, where the underlying structure is carried through in a controlled way. However, the class is neither automatic nor too liberal; it has distinct boundaries and reflects a particular balance that does not endure under arbitrary recombination, as demonstrated by explicit counterexamples to some reverse implications. All things considered, the study offers a useful criterion for determining whether adjoint relations and Drazin-type behavior coincide within a single operator, and it establishes a strong basis for constructing additional examples and creating more precise classifications within operator theory.

References

[1] **V. Wanjala, A. M. Nyongesa**, "On N Quasi D-Operator Operators" *Int. J. Math. And Appl.*, 9(2), 245–248, 2021.

[2] **E. Kreyszig**, *Introductory Functional Analysis with Applications*. Wiley, New York, 1978.

[3] **A. A. S. Jibril**, "On Operators for which $T^{*2}(T)^2 = (T^*T)^2$" *International Mathematical Forum*, 5(46), 2255–2262, 2010.

[4] **V. Revathi, P. M. Naik**, "A study on properties of M quasi-class (Q) operator" *International Journal of Advance Research, Ideas and Innovations in Technology*, 5(5), 387–390, 2019.

[5] **S. D. M. and A. M. K.,** "On the Class of (K-N) Quasi-Normal Operators on Hilbert Space" *Mathematical Theory and Modeling*, 2015, 5(10), 65–68.

[6] **N. Sivakumar and V. Bavithra,** "On the class of (K-N) QUASI-n-NORMAL OPERATORS on HILBERT SPACE" *International Journal of Advance Research, Ideas and Innovations in Technology*, 2016, 2(6), pp. 1–5.

[7] **A. Ben-Israel, T. N. E. Greville**, *Generalized Inverses: Theory and Applications*. 2nd edn., Springer, New York, 2003.

[8] **S. L. Campbell, C. D. Meyer**, *Generalized Inverses of Linear Transformations*. SIAM, Philadelphia, 2009.

[9] **M. Dana, R. Yousefi**, "On the classes of d-normal operators and d-quasi-normal operators on Hilbert space" *Oper. Matrices*, 12(2), 465–487, 2018.

[10] **A. N. Yasser and H. A. Shubber**, "ON (n, m)-D-QUASI OPERATOR ON HILBERT SPACE" *Asia Pacific Journal of Mathematics*, 2026, 13:24, doi: 10.28924/APJM/13-24.

[11] **M. Reed, B. Simon**, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*. Academic Press, 1980.

[12] **Y. H. Saleem, H. A. Shubber**, "The smoothness of schrodinger operator with electromagnetic potential" *European Journal of Pure and Applied Mathematics*, 2019, 12(4), pp. 1441–1454.

[13] **S. H. Jah**, "CLASS OF (A, n)-POWER QUASI-NORMAL OPERATORS IN SEMI-HILBERTIAN SPACES" *International Journal of Pure and Applied Mathematics*, 2014, 93(1), 61–83, doi: 10.12732/ijpam.v93i1.6.

[14] **S. S. A. Hussein, H. A. Shubber**, "Spectral theory of n-normal operators on Hilbert space" *Journal of Interdisciplinary Mathematics*, 2021, 24(7), pp. 1871–1877.

[15] **H. O. AlShammari**, "Higher order (n, m)-Drazin normal operators" *Journal of Inequalities and Applications*, 2024, 2024:18, doi: 10.1186/s13660-024-03095-4.

[16] **S. S. A. Hussein, H. A. Shubber**, "Spectral theory of (n, m)-normal operators on Hilbert space" *Journal of Interdisciplinary Mathematics*, 2021, 24(7), pp. 1907–1915.

[17] **A. Mazin Falih**, "On $(K^* - N)^n$-Quasi Normal Operators" *Journal of Al-Qadisiyah for Computer Science and Mathematics*, 2024, 16(2), pp. Math 131–136, doi: 10.29304/jqcsm.2024.16.21564.