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Bifurcation of Solution in Singularly Perturbed DAEs By Using Lyapunov Schmidt Reduction

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Abstract:
This paper aims to study the bifurcation of solution in singularly perturbed differential algebraic equations (DAEs):

$$x^{\cdot} = f\ (x, y, \alpha, 0)$$

$$0\ = g(x, y, \alpha, 0).$$

Obtained from singularly perturbed ODEs when s approach to 0.

$$x^{\cdot}\ = f\ (x, y, \alpha, \epsilon),$$

$$\epsilon y^{\cdot}\ = g(x, y, \alpha, \epsilon).$$

The main conclusion of this paper is that, under the hypothesis

$$rank D_y g(\mathrm{x}_0, \mathrm{y}_0, \alpha_0, 0)\ =\ m\ -\ 1,$$

the bifurcation of solution in the DAE system will be studied through effect of the system by using Lyapunov Schmidt reduction. Sufficient conditions for the occurrence of some types of bifurcation in the solution are given, such as (Fold, Pitchfork and Transcritical Bifurcation) in *n* dimensional.

Keywords: *DAE, Bifurcation, Singularly Perturbed ODEs, Lyapunov Schmidt Reduction.*

1. INTRODUCTION:

The theory of singularly perturbed is form the mathematical point of view a very inter- esting subject became it is possible to apply with success the results of the more abstract theory of differential equation. There are a lot of scientists and researchers who work on the singularly perturbed ODEs theory.

Ali Nayef. **[2]** in 1981 published" introduction to perturbed techniques and perturbed method in applied mathematics".
Eckhaus, W. **[7]** in 1979 studied the asymptotic analysis of singular perturbation (asymptotic expansions) in terms of a small or a large parameter or coordinating. He started with model simple ordinary equations that can be solved exactly and which progress toward complex partial differential equations.
R. E. O'Malley in **[14]** applied perturbation methods on the nonlinear differential equations problems by providing a new phenomena occururence which has no place in the corresponding linear problems, also explained the major purpose for introducing the per turbation methods on nonlinear problem in order to improve the result in distinctively new phenomena by studying the existence of solutions of periodic problems for all frequencies rather than only a set of characteristic values. His objective was to show the dependence of amplitude on frequency, removal of resonance infinities, and appearance of jump phenomena.
The method of finite dimensional reduction was introduced by Lyapunov **[10]** (1906) and Schmidt **[15]** (1908).

Hale and Sakamoto **[8]** in (1988) applied Lyapunov Schmidt reduction to construct singulary perturbed equilibrium solution to differential equation.
Arnold Neumaier **[3]** in (1991), the lyapunov Schmidt reduction for parameterized equation near singular points.
In **[9]** (2007) the asymptotic stability of an equilibrium solution of the differential algebraic equations (DAEs) is investigated by reducing such DAEs by Liapunov-Schmidt reduction to a corresponding one.

[18] In (2008) the Lyapunov Schmidt reduction for the singularity analysis of finite dimensional is presented
Consider the DAEs:

$$x^{\cdot} = f\,(x, y, \alpha, 0), \tag{1.1}$$

$$0 = g(x, y, \alpha, 0), \tag{1.2}$$

where $(\mathrm{f}, \mathrm{g}) : R^n \times R^m \times R \times R \rightarrow R^n \times R^m$. Define the following related sets:

$$\mathrm{M} = \{(\mathrm{x}, \mathrm{y}, \alpha, 0) \in R^n \times R^m \times R \times R : 0 = \mathrm{g}(\mathrm{x}, \mathrm{y}, \alpha, 0)\}, \tag{1.3}$$

and the set:

$$\mathrm{T} = \mathrm{M} \backslash \mathrm{S}, \tag{1.4}$$

where S is defined by:

$$S = \{(x, y, \alpha, 0) \in M : rankD_y g(x, y, \alpha, 0) = m - 1\}. \tag{1.5}$$

Let $(\mathrm{x}_0, \mathrm{y}_0, \alpha_0, 0) \in M$ such that $f(\mathrm{x}_0, \mathrm{y}_0, \alpha_0, 0) = 0$. If $rankD_y g(\mathrm{x}_0, \mathrm{y}_0, \alpha_0, 0) = m$ then $(\mathrm{x}_0, \mathrm{y}_0, \alpha_0, 0) \in T$ and it is just a non-degenerate equilibrium point. The degenerate equilibrium points belong to the singular surface S that is the points which satisfy the rank condition

$$rankDyg(x, y, \alpha, 0) = m - 1.$$

Since $\frac{\partial g}{\partial y}(x, y, \alpha, 0)$ is singular at singular point $(\mathrm{x}_0, \mathrm{y}_0, \alpha_0, 0)$ the solution may bifurcate at that point, there may be impasse for which the solution does not exist near that point, or the solution is well defined through the singularly. Our study includes the stability of degenerate equilibrium points $(x_0, y_0, \alpha_0, 0) \in S$ of the DAEs for which the solution near that point exists and well is defined. Let $(x_0, y_0, \alpha_0, 0) \in M$ be an equilibrium point for $\alpha = 0$, (i.e), $f(x_0, y_0, \alpha_0, 0) = 0$, and that

$$rank D_y g(x_0, y_0, \alpha_0, 0) = m - 1 \qquad (1.6)$$

The assumption (1.6) states that zero is an eigenvalue of $D_y g(x_0, y_0, \alpha_0, 0)$.

2. Definitions and Concepts

Definition 2.1 [5][12]((m, n)-Fast-Slow System)

System of ordinary differential equations has the form:

$$\frac{dx}{d\tau} = \dot{x} = \epsilon f(x, y, \epsilon), \qquad (2.1)$$

$$\frac{dy}{d\tau} = \dot{y} = g(x, y, \epsilon), \qquad (2.2)$$

is called a m fast-slow system?

Where variable x is called fast variable, variable y is called slow variable. A time-scale decomposition of the singularly perturbed system yields reduced-order representations for the slow and fast subsystems. More specifically in the limit $\epsilon \to 0$ the fast dynamics become instantaneous in the slow time-scale t. By applying the time scale:

setting $t = \tau\epsilon \Rightarrow \tau = \frac{t}{\epsilon}$

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{dx}{dt} = \frac{dx}{d\tau}\frac{1}{\epsilon} = f(x, y, \epsilon)$$

and

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{\epsilon}\frac{dy}{d\tau} = \frac{1}{\epsilon} g(x, y, \epsilon) \Rightarrow \epsilon\frac{dy}{dt} = g(x, y, \epsilon)$$

The gives the equivalent form:

$$\frac{dx}{dt} = \dot{x} = f(x, y, \epsilon) \qquad (2.3)$$

$$\frac{dy}{dt} = \epsilon\dot{y} = g(x, y, \epsilon) \qquad (2.4)$$

the systems (2.3), (2.4) is called n fast-slow system. It refers to t as the fast time scale or fast time and to τ as the slow time scale or slow time.

When s approaches to 0 for system (2.3),(2.4) we get:

$$\dot{x} = f(x, y, 0)$$

$$0 = g(x, y, 0)$$

which represent to DAEs with index one, and it can be readily reduced to an ODEs. Sometimes, one finds that the x variables are slow and the y variables are fast with similar or no changes regarding the notation for the functions f and g.

Definition 2.3 [6] The singularity perturbed ODEs obtained by setting s approaches to 0 on the fast time scale formulation (2.3),(2.4) is called a fast subsystem or fast vector field:

$$x^{\cdot} = f(x, y, 0), \tag{2.7}$$

$$0 = g(x, y, 0). \tag{2.8}$$

The flow of (2.7), (2.8) is called the fast flow.

Bifurcation is a French word that has been introduced into nonlinear dynamics by (Poincare et al. 1899). Bifurcation theory studies the change in behavior of the system with the change in parameters that involves the change in the dynamics behavior. These changes are only qualitative in nature. But there may be changes in situations as well. In bifurcation problems, it is useful to consider a space formed by using the state variables and the control parameters, called the state-control space. Before introduce the definition of bifurcation we will mention to the definition of topological equivalent:

Definition 2.4 [11]

Suppose that $f \in C^1(E_1)$ and $g \in C^1(E_2)$ where E_1 and E_2 are open subsets of R^n. Then the two autonomous systems of differential equations

$$x^{\cdot} = f(x), \tag{2.9}$$

and

$$y^{\cdot} = g(y), \tag{2.10}$$

are said to be topologically equivalent if there is a homeomorphism $H : E_1 \to E_2$ which maps trajectories of (2.9) onto trajectories of (2.10) and preserves their orientation by time.

Definition 2.5 [17] The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.

Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

Definition 2.6 (Bifurcation point [16]) The sudden change in the behavior of the system when a parameter passes through a critical value.

Definition 2.7 (Bifurcation diagram [17]) A bifurcation diagram of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.

Thus, bifurcation is a complex phenomenon occurs in nonlinear systems, it is referring to the branching of solutions at some critical value parameters, which results in a loss of the structural stability and it is one of routes to chaos [1]. Here we will state the bifurcation kinds such as (fold, transcritical and pitchfork bifurcation).

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3. Implicit Function Theorem [4]

This theorem is very important because it represents the beginning point of the basic work in this paper, thus it is convenient to mention it. The Implicit Function Theorem in finite dimensions is concerned with system of equations of the form:

$$f_i(x_1,\cdots,x_n,\alpha_1,\cdots,\alpha_k) = 0, \qquad i = 1,\cdots,n. \tag{3.1}$$

which can be written as vector form where $F : R_n \times R_k \to R_n$.

The Jacobian matrix for (3.2) is:

$$J(x,\alpha) = \left(\frac{\partial f_i}{\partial x_j}(x,\alpha)\right)_{i,j=1,\dots,n} \tag{3.3}$$

We will use a neighborhood of fixed point $(x_0, \alpha_0) \in R^n \times R^k$.

Theorem 3.1 (Implicit Function Theorem [13].)
Suppose that F be given in (3:2), such that:

$$F\,(x_0, \alpha_0) = 0, \qquad (3.4)$$

and

$$detJ\,(x_0,\ \alpha_0) \neq 0. \qquad (3.5)$$

Then there exists neighborhoods U of x_0 in R^n and V of α_0 in R^k and a function : $V \rightarrow U$ such that for every $\alpha \in V$, (3.2) has a unique solution $x = X(\alpha)$ in U. Moreover, if $F \in C^s(R^n)$, then $X \in C^s(R^k)$. In symbols

$$F\,(X(\alpha), \alpha) = 0, \qquad X(\alpha_0) = x_0. \qquad (3.6)$$

---

4. Lyapunov Schmidt reduction of Singularly perturbed ODEs in R^n [9].

Consider singularly parameterized ODEs:

$$x^{\cdot} = f\,(x, y, \alpha, \epsilon), \qquad (4.1)$$

$$\epsilon y^{\cdot} = g(x, y, \alpha, \epsilon), \qquad (4.2)$$

where $f : R^n \times R^m \times R \times R \rightarrow R^n, g : R^n \times R^m \times R \times R \rightarrow R^m, x \in R^n$ is a slow variable, $y \in R^m$ is a fast variable, $0 < \epsilon \ll 1$ and α is the bifurcation parameter. When ϵ approaches to zero we have a parameterized DAEs as follows:

$$x^{\cdot} = f\,(x, y, \alpha, 0), \qquad (4.3)$$

$$0 = g(x, y, \alpha, 0). \qquad (4.4)$$

Suppose that S_0 is a set of all equilibrium points of (4.3),(4.4) it is defined as:

$$S_0 = \{((x^*, y^*, \alpha^*, 0) \in R^n \times R^m \times R \times R : f\,(x^*, y^*, \alpha^*, 0) = g(x^*, y^*, \alpha^*, 0) = 0\},$$

and the critical points of (4.3) and (4.4) should satisfy the constraint condition. Define the function F as follows:

$$F\,(x, v, \alpha, 0) = \begin{bmatrix} f\,(x, v, \alpha, 0), \\ g(x, v, \alpha, 0), \end{bmatrix} \qquad (4.5)$$

In this section Lyapunov Schmidt reduction in R^n will be introduced and we use it to study bifurcation theory on singularly perturbed ODEs when perturbed parameter ϵ approach to zero.
Consider the DAEs:

$$x^{\cdot} = f\,(x, y, \alpha, 0), \qquad (4.6)$$

$$0 = g(x, y, \alpha, 0), \qquad (4.7)$$

such that the rank condition:

$$rankD_y g(x_0, y_0, \alpha_0, 0) = m - 1 \qquad (4.8)$$

is satisfied. Assume that the equilibrium point is $(0, 0, 0, 0)$ and let $D_y g(0, 0, 0, 0) = B$, then from rank condition we have $rank(B)(0, 0, 0, 0) = m - 1$.
Choose complements vector spaces H and N to $kerB$ and $rangeB$ respectively. Then

$$R^m = kerB \oplus H$$

$$R^m = N \oplus rangeB$$

Then we conclude that $dimH = m - 1$ and $dimN = 1$. Define the projections $E : R^m \to rangeB$ and the complementary projection $(I - E) : R^m \to N$ such that the DAEs (4.6),(4.7) expanded to an equivalent pairs of equations

$$x^{\cdot} = f\,(x, y, \alpha, 0), \qquad (4.9)$$
$$0 = Eg(x, y, \alpha, 0), \qquad (4.10)$$

and

$$x^{\cdot} = f\,(x, y, \alpha, 0),$$
$$0 = (I - E)g(x, y, \alpha, 0).$$

Because of this splitting any vector $y \in R^m$ can be decomposed in the form $y = v + w$, where $v \in kerB$ and $w \in H$. Then the equation (4.9), (4.10) can be written as:

$$x^{\cdot} = f\,(x, v + w, \alpha, 0), \qquad (4.11)$$
$$0 = Eg(x, v + w, \alpha, 0). \qquad (4.12)$$

Then in (4.11), (4.12) the second equation can be considered as a map

$$\varphi : R^n \times kerB \times H \times R^r \to rangeB,$$

where

$$\varphi(x, v, w, \alpha, 0) = Eg(x, v + w, \alpha, 0).$$

Now we have:

$$\left(\frac{\partial \mathrm{Eg(x,v + w,\alpha)}}{\partial \mathrm{w}}\right)(0, 0, 0,0) = \mathrm{EB}.$$

Since E act as the identity map on $rangeB$ so

$$\left(\frac{\partial \mathrm{Eg(x,v + w,\alpha)}}{\partial \mathrm{w}}\right)(0, 0, 0,0) = \mathrm{EB}.$$

and since $B : H \to rangeB$ has a full rank at $(0, 0, 0, 0)$, it follows from the implicit function theorem that the second equation of (4.9),(4.10) can be solved uniquely for w near $(0, 0, 0, 0)$, i.e.,
$w = W\,(x, v, \alpha, 0)$, where $W : R^n \times kerB \times R^r \to M$ satisfies

$$Eg(x, v + W\,(x, v, \alpha), \alpha, 0) \equiv 0, \qquad W\,(0, 0, 0, 0) = 0.$$

From the second equation of (4.11), (4.12) and from DAE (4.6), (4.7) we get the reduced DAEs:

$$x^{\cdot} = F\,(x, y, \alpha, 0), \qquad (4.13)$$

$$0 = G(x, y, \alpha, 0), \qquad (4.14)$$

where $(F, G) : R^n \times kerlB \times R^r \to R^n \times N$ defined by:

$$G(x, y, \alpha, 0) = (I - E)g(x, v + W\,(x, v, \alpha), \alpha, 0), \qquad (4.15)$$
$$F\,(x, y, \alpha, 0) = f\,(x, v + W\,(x, v, \alpha), \alpha, 0). \qquad (4.16)$$

Definition 4.1 The equation

$$G(x, v, \alpha, 0) = (I - E)g(x, v + W\,(x, v, \alpha), \alpha, 0) = 0,$$

is called bifurcation equation in one dimensional.

The reduced DAEs equation (4.13), (4.14) has all the information we need from the Liapunov Schmidt. The only disadvantage that it maps the second component y between one dimensional subspaces of R^m. Now the Lyapunov Schmidt reduction will generalize to n-dimensional sub-space.

Consider the ODEs:

$$x^{\cdot} = F\,(x, v, \alpha, \epsilon), \qquad (4.17)$$
$$\epsilon y^{\cdot} = G(x, v, \alpha, \epsilon), \qquad (4.18)$$

when ϵ approach to 0 we get:

$$x^{\cdot} = F(x, v, \alpha, 0), \qquad (4.19)$$
$$0 = G(x, v, \alpha, 0), \qquad (4.20)$$

where

$$(F, G) : R^n \times kerB \times R \to R^n \times N,$$

define by

$$F(x, v, \alpha, 0) = f(x, v + W(x, v, \alpha), \alpha, 0), \qquad (4.21)$$
$$G(x, v, \alpha, 0) = (I - E)g(x, v + W(x, v, \alpha), \alpha, 0), \qquad (4.22)$$

and $(f, g) : R^n \times R^m \times R \times R \to R^n \times R^m$.
Choose explicit coordinate on $kerB$ and N and assume v and v^* be none-zero vectors in $kerB$ and $(rangeB)^{\perp}$ respectively. Then the vector $v \in kerB$ can be uniquely written in the form $v = yv_0$ where $y \in R$ and $v_0 \in kerB$.
Define

$$\breve{G}(x, y, \alpha, 0) = < v_0^{\ *}, G(x, yv_0, \alpha, 0) >,$$

where G is reduced equation (4.23). Now we show that $\breve{G}(x, y, \alpha, 0) = 0$ iff $G(x, yv_0, \alpha, 0) = 0$ so the zero of $\breve{G}$ are one to one correspondence with the solutions of $g(x, y, \alpha, 0) = 0$. Then the function $\breve{G}$ can be written in terms of the original DAEs (4.6),(4.7) by using (4.19), (i.e.)

$$\breve{G}(x, y, \alpha, 0) = < v_0^{\ *}, g(x, yv_0 + W(x, yv_0, \alpha), \alpha, 0). \qquad (4.23)$$

The function $\breve{G}$ is the reduced function to the constraint equation g in the DAEs (4:6),(4.7) in a new change of coordinates. Also the relation between $\breve{G}$ and G is that $\breve{G}$ is just a representation of G in new coordinates. Hence the reduced DAEs in new coordinate are given by

$$x^{\cdot} = \breve{F}(x, y, \alpha, 0), \qquad (4.24)$$
$$0 = \breve{G}(x, y, \alpha, 0), \qquad (4.25)$$

where $(\breve{F}, \breve{G}) : R^n \times R^m \times R \times R \to R^n \times R^m$ such that F˜ defined by

$$\breve{F}(x, y, \alpha, 0) = f(x, yv_0 + W(x, yv_0, \alpha), \alpha, 0), \qquad (4.26)$$

And $\breve{G}$ as defined in (4.23). As we mentioned above $\breve{G}(x, y, \alpha, 0) = 0$ iff $G(x, yv_0, \alpha, 0) = (I - E)g(x, yv_0 + W(x, yv_0, \alpha), \alpha, 0) = 0$.

$$\frac{\partial G}{\partial y}(x, yv_0, \alpha, 0) = (I - E)\frac{\partial g}{\partial y}(x, yv_0 + W(x, yv_0, \alpha), \alpha, 0)\left(v_0 + \frac{\partial W}{\partial y}\right)$$

On evaluating at $(0, 0, 0, 0)$ we have:

$$\frac{\partial G}{\partial y}(0, 0, 0, 0) = (I - E)B\left(v_0 + \frac{\partial W}{\partial y}(0, 0, 0, 0)\right)$$

Since $(I - E)B = 0$, so $\frac{\partial G}{\partial y}(0,0,0,0) = 0$ by similar way we get $\frac{\partial \breve{G}}{\partial y}(0,0,0,0) = 0$. That means the reduced DAEs have a singularly at $(0, 0, 0, 0)$.

Definition 4.2 The equation

$$\breve{G}(x, y, \alpha, 0) = (I - E)g(x, yv_0 + W(x, yv_0, \alpha), \alpha, 0) = 0,$$

is called bifurcation equation in n dimensional.

4.1 Fold bifurcation in R^n

A fold bifurcation point is a pair of equilibrium, meets and disappears with a zero eigenvalue [11]. One of the equilibrium (saddle) is unstable while the other (node) is stable [16]. Now, consider the DAEs (4.24), (4.25), we will study fold bifurcation of the singularly parameterized ODEs system by the following theorem:

Theorem 4.1

Consider the DAEs (4.6),(4.7) defined on S_0 with an equilibrium point $(0,0,0,0)$ and the non-hyperbolic conditions $\frac{\partial f}{\partial x}(0,0,0,0)=0, \frac{\partial g}{\partial x}(0,0,0,0)=0, \frac{\partial^2 g}{\partial x \partial y}(0,0,0,0)=0$ If the following conditions are hold:

1. $< {v_0}^*, \frac{\partial g}{\partial \alpha}(0,0,0,0) > \neq 0$
2. $< {v_0}^*, \frac{\partial^2 g}{\partial x^2}(0,0,0,0)(v, v,) > \neq 0$

Then $(0,0,0,0)$ is a fold bifurcation point for the reduced DAEs (4.13),(4.14) when ϵ approaches to 0.

Proof : Suppose that $(x, y, \alpha, 0) = (0,0,0,0)$ is critical point and consider the reduced DAE(4.13),(4.14) obtained by Lyapunov Schmidt reduction. Differentiate the bifurcation equation (4.14) w.r.t.α we get:

$$\frac{\partial g}{\partial \alpha}(x, y, \alpha, 0) = (I - E)\left[\frac{\partial g}{\partial x}\frac{\partial x}{\partial \alpha}(x, y, \alpha, 0) + \frac{\partial g}{\partial y}\frac{\partial W}{\partial \alpha}(x, y, \alpha, 0) + \frac{\partial g}{\partial \alpha}(x, y, \alpha, 0)\right]$$

Evaluate at (0,0,0,0), we get:

$$\frac{\partial g}{\partial \alpha}(0,0,0,0) = (I - E)\left[\frac{\partial g}{\partial x}\frac{\partial x}{\partial \alpha}(0,0,0,0) + \frac{\partial g}{\partial y}\frac{\partial W}{\partial \alpha}(0,0,0,0) + \frac{\partial g}{\partial \alpha}(0,0,0,0)\right]$$

from rank condition (4.8) we get:

$$\frac{\partial g}{\partial \alpha}(0,0,0,0) = (I - E)\frac{\partial g}{\partial \alpha}(0,0,0,0).$$

Then from condition (1) we get:

$$\frac{\partial G}{\partial \alpha}(0,0,0,0) \neq 0.$$

To prove the second condition we differentiate bifurcation equation (4.14) w.r.t.x twice:

$$\frac{\partial G}{\partial x}(x, y, \alpha, 0) = (I - E)\left[\frac{\partial g}{\partial x}(x, y, \alpha, 0) + \frac{\partial g}{\partial y}\frac{\partial W}{\partial x}(x, y, \alpha, 0) + \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial x}(x, y, \alpha, 0)\right]$$

$$\frac{\partial^2 G}{\partial x^2}(x, y, \alpha, 0) = (I - E)\left[\frac{\partial^2 g}{\partial x^2}(x, y, \alpha, 0) + \frac{\partial^2 g}{\partial x \partial y}(x, y, \alpha, 0)\frac{\partial W}{\partial x}(x, y, \alpha, 0) + \frac{\partial g}{\partial y}(x, y, \alpha, 0)\frac{\partial^2 W}{\partial x^2}(x, y, \alpha, 0)\right].$$

Evaluate at (0,0,0,0) we get:

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) = (I - E)\left[\frac{\partial^2 g}{\partial x^2}(0,0,0,0) + \frac{\partial^2 g}{\partial x \partial y}(0,0,0,0)\frac{\partial W}{\partial x}(0,0,0,0) + \frac{\partial g}{\partial y}(0,0,0,0)\frac{\partial^2 W}{\partial x^2}(0,0,0,0)\right].$$

Then from rank condition (4.8), and condition above given in theorem we get:

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) = (I - E)\left[\frac{\partial^2 g}{\partial x^2}(0,0,0,0)\right]$$

and from condition 2 we see that:

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) \neq 0$$

So the bifurcation equation (4.14) satisfy fold bifurcation conditions.

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5. Transcritical bifurcation in R^n

A transcritical bifurcation is one in which an equilibrium point exists for all values of a parameter and is never destroyed [11]. In transcritical bifurcation there is an exchange of stability between two equilibrium points, there is one unstable and the other is stable equilibrium point. Now we will introduce the transcritical bifurcation theorem for the singularly parameterized ODEs as follows:

Theorem 4.2: Consider the DAEs (4.6),(4.7) defined on S_0 with an equilibrium point $(0,0,0,0)$ and the non-hyperbolic conditions $\frac{\partial f}{\partial x}(0,0,0,0) = 0, \frac{\partial g}{\partial x}(0,0,0,0) = 0, \frac{\partial^2 g}{\partial x \partial y}(0,0,0,0) = 0.$ are satisfied. If the following conditions are hold:

1. $< {v_0}^*, \frac{\partial g}{\partial \alpha}(0,0,0,0) \geq 0$
2. $< {v_0}^*, \frac{\partial^2 g}{\partial x^2}(0,0,0,0)(v, v,) > \neq 0$
3. $< {v_0}^*, \frac{\partial g^2}{\partial \alpha \partial x}(0,0,0,0) v > \neq 0.$

Then $(0,0,0,0)$ is a transcritical bifurcation point for the reduced DAEs (4.13),(4.14) when ϵ approaches to 0.

Proof : Suppose that $(x, y, \alpha, 0) = (0,0,0,0)$ is critical point and consider the reduced DAE(4.13),(4.14) obtained by Lyapunov Schmidt reduction. Differentiate the bifurcation equation (4.14) w.r.t. α as in theorem (4.1) we get:

$$\frac{\partial G}{\partial \alpha}(0,0,0,0) = (I - E)\frac{\partial g}{\partial \alpha}(0,0,0,0).$$

Then from condition (1) we get:

$$\frac{\partial G}{\partial \alpha}(0,0,0,0) = 0.$$

To prove the second condition we differentiate bifurcation equation (4.14) $w.r.t. x$ twice as in theorem (4.1) we get:

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) = (I - E)\left[\frac{\partial^2 g}{\partial x^2}(0,0,0,0)\right],$$

and from condition 2 we see that:

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) \neq 0$$

To prove the third condition we differentiate bifurcation equation (4.14) $w.r.t. x$ and α we get:

$$\frac{\partial^2 G}{\partial x \partial \alpha}(x, y, \alpha, 0) = (I - E)\left[\frac{\partial}{\partial x}\left(\frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial \alpha}(x, y, \alpha, 0) + \frac{\partial g}{\partial y}\frac{\partial W}{\partial \alpha}(x, y, \alpha, 0) + \frac{\partial g}{\partial \alpha}(x, y, \alpha, 0)\right)\right].$$

$\frac{\partial^2 G}{\partial x \partial \alpha}(x,y,\alpha,0) = (I-E)\left[\left(\frac{\partial^2 g}{\partial x^2}\frac{\partial x}{\partial \alpha}(x,y,\alpha,0) + \frac{\partial g}{\partial x}\frac{\partial^2 x}{\partial \alpha^2}(x,y,\alpha,0) + \frac{\partial^2 g}{\partial x \partial y}\frac{\partial W}{\partial \alpha}(x,y,\alpha,0) + \frac{\partial g}{\partial y}\frac{\partial^2 W}{\partial x \partial \alpha}(x,y,\alpha,0) + \frac{\partial^2 g}{\partial x \partial \alpha}(x,y,\alpha,0)\right)\right].$

Evaluate at (0,0,0,0):

$\frac{\partial^2 G}{\partial x \partial \alpha}(0,0,0,0) = (I-E)\left[\left(\frac{\partial^2 g}{\partial x^2}\frac{\partial x}{\partial \alpha}(0,0,0,0) + \frac{\partial g}{\partial x}\frac{\partial^2 x}{\partial \alpha^2}(0,0,0,0) + \frac{\partial^2 g}{\partial x \partial y}\frac{\partial W}{\partial \alpha}(0,0,0,0) + \frac{\partial g}{\partial y}\frac{\partial^2 W}{\partial x \partial \alpha}(0,0,0,0) + \frac{\partial^2 g}{\partial x \partial \alpha}(0,0,0,0)\right)\right].$

Then from rank condition (4.8), and conditions above given in theorem we get:

$$\frac{\partial^2 G}{\partial x \partial \alpha}(0,0,0,0) = (I-E)\left[\frac{\partial^2 g}{\partial x \partial \alpha}(0,0,0,0)\right],$$

and from condition 3 we see that:

$$\frac{\partial^2 \mathrm{G}}{\partial x \partial \alpha}(0,0,0,0) \neq 0$$

So the bifurcation equation (4.14) satisfy transcritical bifurcation conditions.

5.1 **Pitchfork bifurcation in R^n**

In the pitchfork bifurcation, an equilibrium point reverses its stability, and two new equilibrium points are born [11].

Define:

$$f(x,v,\alpha,\epsilon) = xK(x,v,\alpha,\epsilon),$$

$$g(x,v,\alpha,\epsilon) = xU(x,v,\alpha,\epsilon),$$

where $(K,U): R^n \times R^m \times R \times R \rightarrow R^n \times R^m$ and α is the bifurcation parameter ,$x \in R^n$ $y \in R^m$ ϵ approach to 0.

$$F(x,v,\alpha,\epsilon) = \begin{pmatrix} f(x,v,\alpha,\epsilon) \\ g(x,v,\alpha,\epsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $\mathrm{K}(0,\mathrm{v},\alpha,0) = \frac{\partial \mathrm{f}}{\partial \mathrm{x}}$ at $\mathrm{x}=0$, $\mathrm{U}(0,\mathrm{v},\alpha,0) = \frac{\partial \mathrm{g}}{\partial \mathrm{x}}$ at $\mathrm{x}=0$.

Now we will state the pitchfork bifurcation theorem for the DAEs as follows:

Theorem 4.3: Consider the DAEs (4.6),(4.7) defined on S_0 with an equilibrium point (0, 0, 0, 0), and suppose that the non-hyperbolic conditions $\frac{\partial f}{\partial x}(0,0,0,0) = 0, \frac{\partial g}{\partial x}(0,0,0,0) = 0, \frac{\partial^2 g}{\partial x \partial y}(0,0,0,0) = 0$, are satisfied. If the following conditions are hold:

1. $< {v^*}_0, \frac{\partial g}{\partial \alpha}(0,0,0,0) >= 0,$
2. $< {v^*}_0, \frac{\partial^2 g}{\partial x^2}(0,0,0,0)(v,v) >= 0,$
3. $< {v^*}_0, \frac{\partial g^3}{\partial x^3}(0,0,0,0)(v,v,v) > \neq 0, \; < {v^*}_0, \frac{\partial g^2}{\partial \alpha \partial x}(0,0,0,0)v > \neq 0.$

Then $(0,0,0,0)$ is a pitchfork bifurcation point for the reduced DAEs (4.13),(4.14),when ϵ approaches to 0.

Proof : Suppose that $(x,y,\alpha,0) = (0,0,0,0)$ is critical point and consider the reduced DAE(4.13),(4.14) obtained by Lyapunov Schmidt reduction. Form the proof of theorem (4.1) and theorem (4.2) we can see that:

$$\frac{\partial G}{\partial \alpha}(0,0,0,0) = (I-E)\frac{\partial g}{\partial \alpha}(0,0,0,0).$$

Then from condition (1) we get:

$$\frac{\partial G}{\partial \alpha}(0,0,0,0) = 0,$$

and

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) = (I-E)\left[\frac{\partial^2 g}{\partial x^2}(0,0,0,0)\right].$$

and from condition 2 we see that:

$$\frac{\partial^2 G}{\partial x^2}(0,0,0,0) = 0$$

also from the proof of theorem (4.2) we can see that:

$$\frac{\partial^2 G}{\partial x \partial \alpha}(0,0,0,0) = (I-E)\left[\frac{\partial^2 g}{\partial x \partial \alpha}(0,0,0,0)\right],$$

and from condition 3 we see that:

$$\frac{\partial^2 G}{\partial x \partial \alpha}(0,0,0,0) \neq 0.$$

To prove $\frac{\partial^3 G}{\partial x^3}(0,0,0,0) \neq 0$, we differentiate bifurcation equation (4.14) w.r.t.x three times:

$$\frac{\partial^3 G}{\partial x^3}(x,y,\alpha,0)$$
$$= (I-E)\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 g}{\partial x^2}(x,y,\alpha,0) + \frac{\partial^2 g}{\partial x \partial y}(x,y,\alpha,0)\frac{\partial W}{\partial x}(x,y,\alpha,0)\right.\right.$$
$$\left.\left. + \frac{\partial g}{\partial y}(x,y,\alpha,0)\frac{\partial^2 W}{\partial x^2}(x,y,\alpha,0)\right)\right],$$

$$\frac{\partial^3 G}{\partial x^3}(x,y,\alpha,0) = (I-E)\left[\frac{\partial^3 g}{\partial x^3}(x,y,\alpha,0) + \frac{\partial^3 g}{\partial x^2 \partial y}(x,y,\alpha,0)\frac{\partial W}{\partial x}(x,y,\alpha,0) + \right.$$
$$\left.\frac{\partial^2 g}{\partial x \partial y}(x,y,\alpha,0)\frac{\partial^2 W}{\partial x^2}(x,y,\alpha,0) + \frac{\partial^2 g}{\partial x \partial y}(x,y,\alpha,0)\frac{\partial^2 W}{\partial x^2}(x,y,\alpha,0) + \frac{\partial g}{\partial y}(x,y,\alpha,0)\frac{\partial^3 W}{\partial x^3}(x,y,\alpha,0)\right].$$

Then from rank condition (4.8), and from the condition above given in theorem we get:

$$\frac{\partial^3 G}{\partial x^3}(x,y,\alpha,0) = (I-E)\left[\frac{\partial^3 g}{\partial x^3}(x,y,\alpha,0)\right],$$

and from condition 3 we see that:

$$\frac{\partial^3 G}{\partial x^3}(x,y,\alpha,0) \neq 0$$

So the bifurcation equation (4.14) satisfy pitchfork bifurcation conditions. The following example is application of theorem (4.1).

Example 4.1: Consider the singularly parameterized ODEs:
Let

$$(f,g): R^2 \times R^2 \times R \to R^2$$

given by

$$F(x,y,\alpha,\epsilon) = (\epsilon x - y, \alpha + x2 - \epsilon y), x \in R^2, y \in R^2, \alpha \in R, \epsilon \in \mathrm{R}.$$
$$x^{\cdot} = f(x,y,\alpha,\epsilon) = \epsilon x - y,$$
$$\epsilon \dot{y} = g(x,y,\alpha,\epsilon) = \alpha + x^2 - \epsilon x,$$

when ϵ approach to zero we get DAEs:

$$x^{\cdot} = f(x,y,\alpha,0) = -y,$$
$$0 = g(x,y,\alpha,0) = \alpha + x^2,$$

Since $F(0,0,0,0) = 0, D_{x,y}F(0,0,0,0)$ is singular and $\frac{\partial f}{\partial x}(0,0,0,0) \neq 0$ invertible.

So by implicit function theorem we get:

there is an open neighborhood U of (0,0) in R^2 ,and an open neighborhood V of (0) in R, and a map $W : U \to V$ such that $W = W(x,y,\alpha,0)$ so that $W(x,y,\alpha,0)$ is the unique solution in V of the equation

$$x^{\cdot} = f(x,y + W(x,y,\alpha),\alpha,0).$$

Now $(x,y + W(x,y,\alpha),\alpha,0)$ is an equilibrium of F if and only if $g(x,y,\alpha,0) = 0$, where

$$0 = g(x,y + W(x,y,\alpha),\alpha,0).$$

A calculation shows that:

$$\frac{\partial g}{\partial \alpha}(0,0,0,0) = 1 \neq 0$$

and

$$\frac{\partial^2 g}{\partial x^2}(0,0,0,0) = -2 \neq 0$$

Now we have reduced DAEs above. There is a fold bifurcation at the non-hyperbolic critical point (0,0,0,0) at the bifurcation value $\alpha = 0$. If $\alpha = 0$ we get only one equilibrium point which is (0,0,0,0). If $\alpha > 0$ then there are two equilibrium points which are $\sqrt{(\mp\alpha, 0)}$. If $\alpha < 0$ then there are no equilibrium points at all. Hence the DAEs satisfies the non-hyperbolic conditions $\frac{\partial f}{\partial x}(0,0,0,0) = 0, \frac{\partial g}{\partial x}(0,0,0,0) = 0$, our DAEs satisfies the fold bifurcation conditions above.[11]

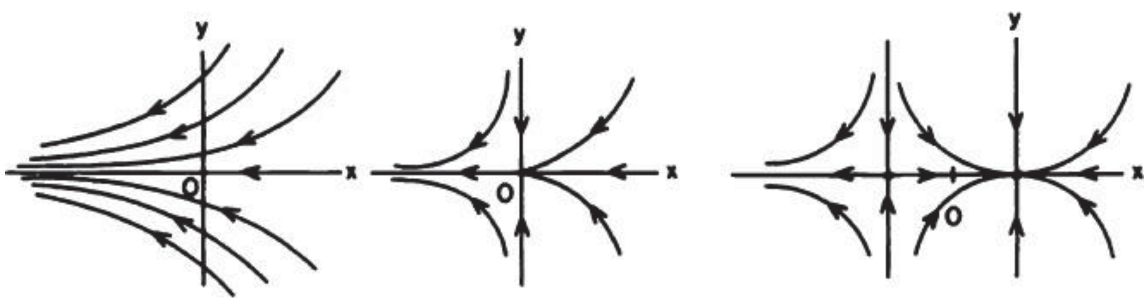

Figure 1: Bifurcation diagrams for the Fold Bifurcation. From left to right, ($\alpha < 0, \alpha = 0$ and $\alpha > 0$.).

References:

1. A. Panfilov, S. Maree (2005). Non-linear dynamical systems, Utrecht University, Utrecht.

2. Ali. Nayfeh (1981). Introduction to perturbation Techniques, Wiley, New York,
3. A. Neumaier (1991). Generalized Lyapuvon Schmidt reduction for parameterized equa- tions at near singular points, Institut fu¨r Mathematik, Universita¨t Wien Strudlhofgasse 4, A-1090 Wien, asturia, MSC Classification:58C15.
4. B. G. Celayeta (1998). Stability for Differential-Algebraic Equations, PhD Thesis, Universidad Publica de Navarra.
5. Kuehn (2015). Multiple Time Scale Dynamics, Springer Cham Heidelberg New York Dordrecht London.
6. E.M. De-Jager (1996). The Theory of Singular Perturbations, University of Amsterdam, The Netherlands.
7. Eckhaus W. (1979). Asymptotic Analysis of Singular Perturbations, North- Holland, Amsterdam.